SUT Journal of Mathematics
Vol. 38, No. 1 (2002), 83–126

Hochschild cohomology ring of the integral group ring of the generalized quaternion group

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(Received March 25, 2002; Revised May 16, 2002)

Abstract. We determine the ring structure of the Hochschild cohomology $HH^*(\mathbb{Z}Q_t)$ of the integral group ring of the generalized quaternion group Q_t.

AMS 1991 *Mathematics Subject Classification.* 16E40, 20J06.

Key words and phrases. Hochschild cohomology, cohomology ring, quaternion group, cup product.

Introduction

The cohomology theory of associative algebras was initiated by Hochschild [Hoch], Cartan and Eilenberg [CaE] and MacLane [M]. Let R be a commutative ring and Λ an R-algebra which is a finitely generated projective R-module. If M is a Λ-bimodule (i.e., a $\Lambda \otimes_R \Lambda^{\mathrm{op}}$-module), then the n-th Hochschild cohomology of Λ with coefficients in M is defined by

$$H^n(\Lambda, M) := \mathrm{Ext}^n_{\Lambda\otimes_R\Lambda^{\mathrm{op}}}(\Lambda, M).$$

Cup product gives $HH^*(\Lambda) := \bigoplus_{n\geqslant 0} HH^n(\Lambda)$ a graded ring structure with identity $1 \in Z\Lambda \simeq HH^0(\Lambda)$, where $HH^n(\Lambda)$ denotes $H^n(\Lambda,\Lambda)$ and $Z\Lambda$ denotes the center of Λ. However, to determine the multiplicative structure of the Hochschild cohomology ring is generally difficult task.

We are interested in the case $\Lambda = RG$ for a finite group G. As for the additive structure of the Hochschild cohomology, it was well known that $HH^n(RG)$ is isomorphic to the direct sum of the ordinary group cohomology of the centralizers of representatives of the conjugacy classes of G (see [B, Theorem 2.11.2], [SiW, Section 4]):

$$HH^*(RG) \simeq \bigoplus_j H^*(G_j, R).$$

As for the multiplicative structure of $HH^*(RG)$, when G is a finite abelian group, Holm [Hol] and Cibils and Solotar [CiSo] prove the following ring isomorphism exists:

$$HH^*(RG) \simeq RG \otimes_R H^*(G, R).$$

If G is a non abelian group, it seems more difficult to investigate the ring structure of $HH^*(RG)$. However, Siegel and Witherspoon [SiW] define a new product on $\bigoplus_j H^*(G_j, R)$, making the above additive isomorphism multiplicative. Besides, they calculate the Hochschild cohomology rings of $\mathbb{F}_3 S_3, \mathbb{F}_2 A_4$, $\mathbb{F}_2 D_{2^n}$ using this new product. In this paper, we calculate the ring structure $HH^*(\mathbb{Z}Q_t)$ for arbitrary generalized quaternion group Q_t by the method different from [SiW].

We will describe the detail. Let $Q_t = \langle x, y | x^{2t} = 1, x^t = y^2, yxy^{-1} = x^{-1} \rangle$ be the generalized quaternion group of order $4t$ for any positive integer $t \geqslant 2$. We set $\Lambda = \mathbb{Z}Q_t$. In this paper we determine the ring structure of the Hochschild cohomology $HH^*(\Lambda) := \bigoplus_{n \geqslant 0} HH^n(\Lambda)$ for $t \geqslant 2$ using a ring isomorphism

$$HH^*(\Lambda) \xrightarrow{\sim} H^*(G, {}_\varphi\Lambda) := \bigoplus_{n \geqslant 0} H^n(G, {}_\varphi\Lambda)$$

and calculating the ordinary cup product in $H^*(G, {}_\varphi\Lambda)$ above. In fact, Q_t has the well known periodic resolution of period 4. To compute the ordinary cup product, we use a diagonal approximation on the periodic resolution of Q_t stated in [HaSa].

In Section 1, as preliminaries, we describe some definitions and properties about Hochschild cohomology, and that the isomorphism $HH^n(\Lambda) \xrightarrow{\sim} H^n(G, {}_\varphi\Lambda)$ preserves the cup products.

In Section 2, we obtain particular generators of $H^n(Q_t, {}_\varphi\Lambda)$ as a $\mathbb{Z}$-module (Proposition 1). In fact, although the module structure of $H^n(Q_t, {}_\varphi\Lambda)$ is easily obtained by its additive decomposition (Lemma 1), we need the particular generators to determine the ring structure of $H^*(Q_t, {}_\varphi\Lambda)$ in Section 3.

We calculate the cup product for the generators of $H^*(Q_t, {}_\varphi\Lambda)$ for t even (resp. t odd) in Section 3.1 (resp. Section 3.2). In Section 3.3, as the main theorem of this paper, we give an explicit description of the cohomology ring $H^*(Q_t, {}_\varphi\Lambda)$ summarizing the calculations in Sections 3.1 and 3.2, which means that the ring structure of $HH^*(\Lambda)$ is completely determined.

§1. Preliminaries

Let R be a commutative ring and Λ an R-algebra which is a finitely generated projective R-module. If M is a Λ-bimodule (i.e., a $\Lambda \otimes_R \Lambda^{\mathrm{op}}$-module), then the n-th Hochschild cohomology of Λ with coefficients in M is defined by

$$H^n(\Lambda, M) := \mathrm{Ext}^n_{\Lambda\otimes_R \Lambda^{\mathrm{op}}}(\Lambda, M).$$

Suppose N is another Λ-bimodule. Then for every pair of integers $p, q \geqslant 0$ there is a (Hochschild) cup product

$$H^p(\Lambda, M) \otimes_R H^q(\Lambda, N) \xrightarrow{\smile} H^{p+q}(\Lambda, M \otimes_\Lambda N).$$

If we put $M = N = \Lambda$, then the cup product gives $HH^*(\Lambda) := \bigoplus_{n\geqslant 0} HH^n(\Lambda)$ the structure of a graded ring with identity $1 \in Z\Lambda \simeq HH^0(\Lambda)$, where $HH^n(\Lambda)$ denotes $H^n(\Lambda, \Lambda)$ and $Z\Lambda$ denotes the center of Λ. $HH^*(\Lambda)$ is called the Hochschild cohomology ring of Λ. Moreover, the Hochschild cohomology ring $HH^*(\Lambda)$ is anti-commutative, that is, for $\alpha \in HH^p(\Lambda)$ and $\beta \in HH^q(\Lambda)$ we have $\alpha\beta = (-1)^{pq}\beta\alpha$ (see [Sa2, Proposition 1.2] for example).

Let G be a finite group. We put $\Lambda = RG$. The following isomorphism is well known:

$$H^n(\Lambda, M) \xrightarrow{\sim} H^n(G, {}_\varphi M) := \mathrm{Ext}^n_\Lambda(R, {}_\varphi M).$$

In the above, ${}_\varphi M$ denotes M regarded as a G-module using a ring homomorphism $\varphi : \Lambda \to \Lambda \otimes_R \Lambda^{\mathrm{op}}; x \mapsto x \otimes (x^{-1})^\circ$ for $x \in G$. $H^n(G, {}_\varphi M)$ denotes the ordinary n-th group cohomology.

Suppose A and B are G-modules. Then for every pair of integers $p, q \geqslant 0$ there exists a homomorphism called (ordinary) cup product

$$H^p(G, A) \otimes_R H^q(G, B) \xrightarrow{\smile} H^{p+q}(G, A \otimes_R B).$$

Note that the isomorphism stated above preserves cup products, that is, the following diagram is commutative for Λ-bimodules A and B:

$$\begin{array}{ccc}
H^p(\Lambda, A) \otimes_R H^q(\Lambda, B) & \xrightarrow{\smile} & H^{p+q}(\Lambda, A \otimes_\Lambda B) \\
\wr\big\downarrow & & \big\downarrow\wr \\
H^p(G, {}_\varphi A) \otimes_R H^q(G, {}_\varphi B) & \xrightarrow[\smile_\mu]{} & H^{p+q}\left(G, {}_\varphi(A \otimes_\Lambda B)\right).
\end{array}$$

In the above, $\smile_\mu$ denotes the map induced by the (ordinary) cup product and a left Λ-homomorphism $\mu : {}_\varphi A \otimes_R {}_\varphi B \to {}_\varphi(A \otimes_\Lambda B);\ a \otimes_R b \mapsto a \otimes_\Lambda b$. If we put $A = B = \Lambda$ and identify Λ with $\Lambda \otimes_\Lambda \Lambda$ as a Λ-bimodule, then we have a ring isomorphism

$$HH^*(\Lambda) \xrightarrow{\sim} H^*(G, {}_\varphi\Lambda) := \bigoplus_{n\geqslant 0} H^n(G, {}_\varphi\Lambda)$$

(*cf.* [SiW, Proposition 3.2], [Sa3, Section 1] or [NSa]). In the following, we write $\otimes$ in place of $\otimes_R$ for brevity.

§2. Module structure

Let Q_t denote the generalized quaternion group of order $4t$ for any positive integer $t \geqslant 2$:

$$Q_t = \langle x, y | x^{2t} = 1, x^t = y^2, yxy^{-1} = x^{-1} \rangle.$$

In the following, we denote $\mathbb{Z}Q_t$ by Λ. In this section, we determine the generators of $H^n(Q_t, {}_\varphi\Lambda)$. In fact, although the module structure of $H^n(Q_t, {}_\varphi\Lambda)$ is easily obtained (Lemma 1), to determine the ring structure of $H^*(Q_t, {}_\varphi\Lambda)$ in Section 3, we need particular generators of $H^n(Q_t, {}_\varphi\Lambda)$ as a $\mathbb{Z}$-module.

2.1. A resolution of Q_t and lemmas.

The following periodic Λ-free resolution of $\mathbb{Z}$ of period 4 is well known (see [CaE, Chapter XII, Section 7], [T, Chapter 3, Periodicity]):

$$\begin{aligned}
(Y, \delta): \quad & \cdots \to \Lambda^2 \xrightarrow{\delta_1} \Lambda \xrightarrow{\delta_4} \Lambda \xrightarrow{\delta_3} \Lambda^2 \xrightarrow{\delta_2} \Lambda^2 \xrightarrow{\delta_1} \Lambda \xrightarrow{\varepsilon} \mathbb{Z} \to 0, \\
& \delta_1(c_1, c_2) = c_1(x-1) + c_2(y-1), \\
& \delta_2(c_1, c_2) = \left(c_1 L + c_2(xy+1), -c_1(y+1) + c_2(x-1)\right), \\
& \delta_3(c) = (c(x-1), -c(xy-1)), \\
& \delta_4(c) = cN,
\end{aligned}$$

where L denotes $x^{t-1} + x^{t-2} + \cdots + 1$ $(\in \Lambda)$, Λ^2 denotes the direct sum $\Lambda \oplus \Lambda$ and N denotes $\sum_{w \in Q_t} w$ $(\in \Lambda)$. Applying the functor $\operatorname{Hom}_\Lambda(-, {}_\varphi\Lambda)$ to the periodic resolution (Y, δ), we have the following complex which gives $H^n(Q_t, {}_\varphi\Lambda)$, where we identify $\operatorname{Hom}_\Lambda(Y_0, {}_\varphi\Lambda)$ with Λ, $\operatorname{Hom}_\Lambda(Y_1, {}_\varphi\Lambda)$ with Λ^2 and so on:

$$\begin{aligned}
& \left(\operatorname{Hom}_\Lambda(Y, {}_\varphi\Lambda), \delta^\#\right) : 0 \to \Lambda \xrightarrow{\delta_1^\#} \Lambda^2 \xrightarrow{\delta_2^\#} \Lambda^2 \xrightarrow{\delta_3^\#} \Lambda \xrightarrow{\delta_4^\#} \Lambda \xrightarrow{\delta_1^\#} \cdots, \\
& \delta_1^\#(\lambda) = ((x-1)\lambda, (y-1)\lambda), \\
& \delta_2^\#(\lambda_1, \lambda_2) = (L\lambda_1 - (y+1)\lambda_2, (xy+1)\lambda_1 + (x-1)\lambda_2), \\
& \delta_3^\#(\lambda_1, \lambda_2) = (x-1)\lambda_1 - (xy-1)\lambda_2, \\
& \delta_4^\#(\lambda) = N\lambda.
\end{aligned}$$

To give explicit generators of $H^n(Q_t, {}_\varphi\Lambda)$ as a $\mathbb{Z}$-module in Section 2.2, we prove the following two lemmas.

Lemma 1 (Additive decomposition). *The Hochschild cohomology $H^n(Q_t, {}_\varphi\Lambda)$ $(\simeq HH^n(\Lambda))$ is given as follows:*

$H^n(Q_t, {}_\varphi\Lambda)$

$$
= \begin{cases} \mathbb{Z}^{t+3} & \textit{for} \quad n = 0 \\ (\mathbb{Z}/4t)^2 \oplus (\mathbb{Z}/2t)^{t-1} \oplus (\mathbb{Z}/4)^2 & \textit{for} \quad n \equiv 0 \mod 4,\ n \neq 0 \\ 0 & \textit{for} \quad n \equiv 1 \mod 4 \\ \begin{cases} (\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/2t)^{t-1} \oplus (\mathbb{Z}/4)^2 & \\ (\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/2t)^{t-1} \oplus (\mathbb{Z}/4)^2 & \end{cases} & \begin{array}{l} \textit{for} \quad n \equiv 2 \mod 4, t \textit{ even} \\ \textit{for} \quad n \equiv 2 \mod 4, t \textit{ odd} \end{array} \\ 0 & \textit{for} \quad n \equiv 3 \mod 4. \end{cases}
$$

Proof. Let G be a finite group and R a commutative ring. It is well known that the Hochschild cohomology of the group ring RG is isomorphic to the direct sum of the cohomology of the centralizers G_j of representatives of the conjugacy classes of G (see [B, Theorem 2.11.2], [SiW, Section 4]):

$$
H^n(G, {}_{\varphi}RG) \cong \bigoplus_j H^n(G_j, R).
$$

Now we put $G = Q_t$ and $R = \mathbb{Z}$. The $t + 3$ conjugacy classes of Q_t are given as follows:

$$
\begin{gathered} \{1\}, \quad \{x^t\}, \quad \{x^i, x^{-i}\} \ \text{(for } 1 \leqslant i \leqslant t-1), \\ \{x^m y \mid m \geqslant 0 \text{ even}\}, \quad \{x^m y \mid m \geqslant 1 \text{ odd}\}. \end{gathered}
$$

If we take representatives $1, x^t, x^i$ (for $1 \leqslant i \leqslant t-1$), y, xy from each of them, then the centralizers of them are given by Q_t, Q_t, $\langle x \rangle$ (for $1 \leqslant i \leqslant t-1$), $\langle y \rangle$, $\langle xy \rangle$, respectively. Note that $\langle x \rangle$, $\langle y \rangle$ and $\langle xy \rangle$ are cyclic groups of order $2t$, 4 and 4, respectively. We know the following (see [T, Chapter 3]):

$$
H^n(Q_t, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for} \quad n = 0 \\ \mathbb{Z}/4t & \text{for} \quad n \equiv 0 \mod 4 \text{ and } n \neq 0 \\ 0 & \text{for} \quad n \equiv 1 \mod 4 \\ \begin{cases} (\mathbb{Z}/2)^2 \\ \mathbb{Z}/4 \end{cases} & \begin{array}{l} \text{for} \quad n \equiv 2 \mod 4 \text{ and } t \text{ even} \\ \text{for} \quad n \equiv 2 \mod 4 \text{ and } t \text{ odd} \end{array} \\ 0 & \text{for} \quad n \equiv 3 \mod 4, \end{cases}
$$

$$
H^n(C_k, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for} \quad n = 0 \\ \mathbb{Z}/k & \text{for} \quad n \equiv 0 \mod 2 \text{ and } n \neq 0 \\ 0 & \text{for} \quad n \equiv 1 \mod 2, \end{cases}
$$

where C_k denotes the cyclic group of order k. Thus we have the results. $\square$

In the following, we give some preliminaries for the change of basis which will be used in the proof of Proposition 1. For any integer $0 \leqslant i \leqslant t+1$, we set

$$
M_i = \begin{cases} 1 + x^2 + x^4 + \cdots + x^{2t-2} & (i = 0) \\ x^i + x^{i+2} + x^{i+4} + \cdots + x^{2t-i} & (1 \leqslant i \leqslant t-1) \\ x^t & (i = t) \\ 0 & (i = t+1). \end{cases}
$$

In particular, M denotes M_0. It is easy to see that the following equations hold:

$$
\begin{aligned}
Mx^i &= \begin{cases} M & (i \text{ even}) \\ Mx & (i \text{ odd}), \end{cases} \\
yM &= My, \\
M &= M_2 + 1, \\
M_{i+2} &= M_i - \left(x^i + x^{-i}\right) \qquad\qquad (i = 1, 2, \dots, t-1), \\
M_{2t-i} &= M_{2t-i+2} + x^i + x^{-i} \quad (i = t+1, t+2, \dots, 2t-1).
\end{aligned}
$$

Furthermore, as elements of Λ^2, we set

$$
\boldsymbol{a}_{2i} = \begin{cases}
(y, y) & i = 0 \\
\left(M_{2i+2}y, x^{2i+2}y\right) & \begin{cases} i = 1, 2, \dots, \frac{t}{2} - 1 & (t \text{ even}) \\ i = 1, 2, \dots, \frac{t-1}{2} & (t \text{ odd}) \end{cases} \\
\left(-M_{2t-2i}y, x^{2i+2}y\right) & \begin{cases} i = \frac{t}{2}, \frac{t}{2} + 1, \dots, t-2 & (t \text{ even}) \\ i = \frac{t+1}{2}, \frac{t+3}{2}, \dots, t-2 & (t \text{ odd}) \end{cases} \\
(-My, 0) & i = t-1 \\
(-y, x^2y) & i = t,
\end{cases}
$$

$$
\boldsymbol{a}_{2i+1} = \begin{cases}
(0, xy) & i = 0 \\
\left(M_{2i+1}y, x^{2i+1}y\right) & \begin{cases} i = 1, 2, \dots, \frac{t}{2} & (t \text{ even}) \\ i = 1, 2, \dots, \frac{t-1}{2} & (t \text{ odd}) \end{cases} \\
\left(-M_{2t-(2i-1)}y, x^{2i+1}y\right) & \begin{cases} i = \frac{t}{2} + 1, \frac{t}{2} + 2, \dots, t-1 & (t \text{ even}) \\ i = \frac{t+1}{2}, \frac{t+3}{2}, \dots, t-1 & (t \text{ odd}) \end{cases} \\
(Mxy, 0) & i = t,
\end{cases}
$$

$$
\boldsymbol{b}_i = \begin{cases}
-\left(\left(x^i + x^{-i}\right)y, \left(x^i - x^{i+2}\right)y\right) & i = 0, 1, \dots, 2t-1 \\
\left(My, \left(1 + x^2\right)y\right) & i = 2t \\
(Mxy, 2xy) & i = 2t+1,
\end{cases}
$$

$$
\boldsymbol{b}'_{2i} = \left(My, \left(x^{-2i} + x^{2i+2}\right)y\right) \quad \begin{cases} i = 1, 2, \dots, \frac{t}{2} - 1 & (t \text{ even}) \\ i = 1, 2, \dots, \frac{t-1}{2} & (t \text{ odd}), \end{cases}
$$

$$
\boldsymbol{b}'_{2i-1} = \left(Mxy, \left(x^{1-2i} + x^{2i+1}\right)y\right) \quad \begin{cases} i = 1, 2, \dots, \frac{t}{2} & (t \text{ even}) \\ i = 1, 2, \dots, \frac{t-1}{2} & (t \text{ odd}). \end{cases}
$$

It is easy to see that $\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{2t+1}$ are linearly independent on $\mathbb{Z}$ by comparing the second components of $\boldsymbol{a}_i$'s.

Lemma 2. *The following equations hold in Λ^2:*

$$\text{(i)}\quad \boldsymbol{a}_{2i} = \begin{cases} -\boldsymbol{a}_0 + \sum_{j=1}^{i} \boldsymbol{b}_{2j} + \boldsymbol{b}_{2t} & i = 1, 2, \dots, t-2 \\ -2\boldsymbol{a}_0 + \sum_{j=1}^{t-1} \boldsymbol{b}_{2j} + \boldsymbol{b}_{2t} & i = t-1 \\ -3\boldsymbol{a}_0 + \sum_{j=1}^{t-1} \boldsymbol{b}_{2j} + 2\boldsymbol{b}_{2t} & i = t, \end{cases}$$

$$\text{(ii)}\quad \boldsymbol{a}_{2i+1} = \begin{cases} -\boldsymbol{a}_1 + \sum_{j=0}^{i-1} \boldsymbol{b}_{2j+1} + \boldsymbol{b}_{2t+1} & i = 1, 2, \dots, t-1 \\ -2\boldsymbol{a}_1 + \boldsymbol{b}_{2t+1} & i = t, \end{cases}$$

$$\text{(iii)}\quad \boldsymbol{b}_0 = \sum_{i=1}^{t-1} \boldsymbol{b}_{2i} + 2\boldsymbol{b}_{2t} - 4\boldsymbol{a}_0,$$

$$\text{(iv)}\quad \boldsymbol{b}_{2t-1} = -\sum_{i=1}^{t-1} \boldsymbol{b}_{2i-1} - 2\boldsymbol{b}_{2t+1} + 4\boldsymbol{a}_1,$$

$$\text{(v)}\quad \boldsymbol{b}'_{2i} = \sum_{j=1}^{i} \boldsymbol{b}_{2j} - \sum_{j=1}^{i} \boldsymbol{b}_{2t-2j} + \boldsymbol{b}_{2t} \qquad \begin{cases} i = 1, 2, \dots, \frac{t}{2} - 1 & (t \textit{ even}) \\ i = 1, 2, \dots, \frac{t-1}{2} & (t \textit{ odd}), \end{cases}$$

$$\text{(vi)}\quad \boldsymbol{b}'_{2i+1} = \sum_{j=0}^{i} \boldsymbol{b}_{2j+1} - \sum_{j=0}^{i} \boldsymbol{b}_{2t-(2j+1)} + \boldsymbol{b}_{2t+1}$$

$$\begin{cases} i = 0, 1, \dots, \frac{t}{2} - 1 & (t \textit{ even}) \\ i = 0, 1, \dots, \frac{t-3}{2} & (t \textit{ odd}). \end{cases}$$

Moreover, there exist $(t+1) \times (t+1)$ matrices P_1, P_2 satisfying $\det P_1 = \det P_2 = 1$ *such that*

$$(\boldsymbol{a}_0\ \boldsymbol{a}_2\ \dots\ \boldsymbol{a}_{2t}) = (\boldsymbol{a}_0\ \boldsymbol{b}_2\ \boldsymbol{b}_4\ \dots\ \boldsymbol{b}_{2t})\, P_1,$$
$$(\boldsymbol{a}_1\ \boldsymbol{a}_3\ \dots\ \boldsymbol{a}_{2t+1}) = (\boldsymbol{a}_1\ \boldsymbol{b}_1\ \boldsymbol{b}_3\ \dots \boldsymbol{b}_{2t-3}\ \boldsymbol{b}_{2t+1})\, P_2.$$

Proof. Straightforward. From (i) and (ii), we obtain

$$P_1 = \begin{pmatrix} 1 & -1 & -1 & \dots & \dots & -1 & -2 & -3 \\ 0 & 1 & 1 & \dots & \dots & 1 & 1 & 1 \\ 0 & 0 & 1 & \dots & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \ddots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & & \ddots & 1 & 1 & 1 \\ 0 & 0 & 0 & \dots & \dots & 0 & 1 & 1 \\ 0 & 1 & 1 & \dots & \dots & 1 & 1 & 2 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 1 & -1 & -1 & \dots & \dots & -1 & -1 & -2 \\ 0 & 1 & 1 & \dots & \dots & 1 & 1 & 0 \\ 0 & 0 & 1 & \dots & \dots & 1 & 1 & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & & \ddots & 1 & 1 & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 & 1 & 0 \\ 0 & 1 & 1 & \dots & \dots & 1 & 1 & 1 \end{pmatrix}.$$

Hence we have $\det P_1 = \det P_2 = 1$. This completes the proof. □

Lemma 2 shows that $\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \dots, \boldsymbol{b}_{2t-2}, \boldsymbol{b}_{2t}, \boldsymbol{b}_{2t+1}$ are linearly independent on $\mathbb{Z}$ and so we have the following equality:

$$\bigoplus_{i=0}^{2t+1} \mathbb{Z}\boldsymbol{a}_i = \mathbb{Z}\boldsymbol{a}_0 \oplus \mathbb{Z}\boldsymbol{a}_1 \oplus \bigoplus_{i=1}^{2t-2} \mathbb{Z}\boldsymbol{b}_i \oplus \mathbb{Z}\boldsymbol{b}_{2t} \oplus \mathbb{Z}\boldsymbol{b}_{2t+1}. \tag{2.1}$$

2.2. Explicit generators of $H^n(Q_t, {}_{\varphi}\Lambda)$.

Next, we give particular generators of $H^n(Q_t, {}_{\varphi}\Lambda)$ using Lemmas 1 and 2. The generators of $H^n(Q_t, {}_{\varphi}\Lambda)$ will be used in the calculation of the cohomology ring $H^*(Q_t, {}_{\varphi}\Lambda)$ in Section 3.

Proposition 1. *The module structure of* $H^n(Q_t, {}_{\varphi}\Lambda)$ *is represented by the form of the subquotient of the complex* $\mathrm{Hom}_{\Lambda}(Y, {}_{\varphi}\Lambda)$ *as follows:*

$H^n(Q_t, {}_{\varphi}\Lambda)$

$$
= \begin{cases}
\mathbb{Z} \oplus \mathbb{Z}x^t \oplus \bigoplus_{i=1}^{t-1} \mathbb{Z}\left(x^i + x^{-i}\right) \oplus \mathbb{Z}My \oplus \mathbb{Z}Mxy & \textit{for } n = 0, \\
\begin{aligned} &\mathbb{Z}/4t \oplus \mathbb{Z}x^t/4t \ \oplus \bigoplus_{i=1}^{t-1} \mathbb{Z}\left(x^i + x^{-i}\right)/2t \\ &\qquad\oplus \mathbb{Z}My/4 \oplus \mathbb{Z}Mxy/4 \end{aligned} & \textit{for } n \equiv 0 \mod 4,\ n \neq 0, \\
0 & \textit{for } n \equiv 1 \mod 4, \\
\begin{cases}
\begin{aligned} &\mathbb{Z}(1,0)/2 \oplus \mathbb{Z}(0,1)/2 \oplus \mathbb{Z}(x^t,0)/2 \\ &\oplus \mathbb{Z}(0,x^t)/2 \oplus \bigoplus_{i=1}^{t-1} \mathbb{Z}\left(x^i,0\right)/2t \\ &\oplus \mathbb{Z}(y,y)/4 \oplus \mathbb{Z}(0,xy)/4 \end{aligned} & \textit{for } n \equiv 2 \mod 4,\ t \textit{ even}, \\
\begin{aligned} &\mathbb{Z}\left(\tfrac{t-1}{2}, 1\right)/4 \oplus \mathbb{Z}\left(\tfrac{t-1}{2}x^t, x^t\right)/4 \\ &\oplus \bigoplus_{i=1}^{t-1} \mathbb{Z}\left(x^i,0\right)/2t \oplus \mathbb{Z}(y,y)/4 \\ &\oplus \mathbb{Z}(0,xy)/4 \end{aligned} & \textit{for } n \equiv 2 \mod 4,\ t \textit{ odd},
\end{cases} \\
0 & \textit{for } n \equiv 3 \mod 4.
\end{cases}
$$

In the above, M/m denotes the quotient module M/mM for a $\mathbb{Z}$-module M and an element $m \in \mathbb{Z}$.

Proof. Let $\lambda = \sum_{i=0}^{2t-1} \sum_{j=0}^{1} z_{i,j}x^iy^j$ $(z_{i,j} \in \mathbb{Z})$ be any element of ${}_{\varphi}\Lambda$. By the action of Q_t on ${}_{\varphi}\Lambda$, we have

$$
\begin{aligned}
x\lambda &= \sum_{i=0}^{2t-1} z_{i,0}x^i + \sum_{i=0}^{2t-1} z_{i,1}x^{i+2}y \\
&= \sum_{i=0}^{2t-1} z_{i,0}x^i + z_{2t-2,1}y + z_{2t-1,1}xy + \sum_{i=2}^{2t-1} z_{i-2,1}x^iy, \\
y\lambda &= \sum_{i=0}^{2t-1} z_{i,0}x^{-i} + \sum_{i=0}^{2t-1} z_{i,1}x^{-i}y \\
&= z_{0,0} + \sum_{i=1}^{2t-1} z_{2t-i,0}x^i + z_{0,1}y + \sum_{i=1}^{2t-1} z_{2t-i,1}x^iy, \\
xy\lambda &= \sum_{i=0}^{2t-1} z_{i,0}x^{-i} + \sum_{i=0}^{2t-1} z_{i,1}x^{2-i}y \\
&= z_{0,0} + \sum_{i=1}^{2t-1} z_{2t-i,0}x^i + \sum_{i=0}^{2} z_{2-i,1}x^iy + \sum_{i=3}^{2t-1} z_{2t-i+2,1}x^iy.
\end{aligned}
$$

In the following, we calculate the cohomology in each cases of n modulo 4.

(i) The case $n = 0$: $H^0(Q_t, {}_\varphi\Lambda) = \mathrm{Ker}\ \delta_1^{\#} = \mathbb{Z} \oplus \mathbb{Z}x^t \oplus \bigoplus_{i=1}^{t-1} \mathbb{Z}\left(x^i + x^{-i}\right) \oplus \mathbb{Z}My \oplus \mathbb{Z}Mxy$ holds. In fact, for $\lambda = \sum_{i=0}^{2t-1}\sum_{j=0}^{1} z_{i,j}x^iy^j \in \Lambda$, we have

$$\begin{aligned}
\lambda \in \mathrm{Ker}\ \delta_1^{\#} &\iff \begin{cases} (x-1)\lambda = 0 \\ (y-1)\lambda = 0 \end{cases} \\
&\iff \begin{cases} \displaystyle\sum_{i=0}^{1}\left(z_{2t+i-2,1} - z_{i,1}\right)x^iy + \sum_{i=2}^{2t-1}\left(z_{i-2,1} - z_{i,1}\right)x^iy = 0 \\ \displaystyle\sum_{i=1}^{2t-1}\left(z_{2t-i,0} - z_{i,0}\right)x^i + \sum_{i=1}^{2t-1}\left(z_{2t-i,1} - z_{i,1}\right)x^iy = 0 \end{cases} \\
&\iff \begin{cases} z_{i,0} = z_{2t-i,0} \quad (i = 1, 2, \dots, t-1) \\ z_{i,1} = z_{i+2,1} \quad (i = 0, 1, \dots, 2t-3) \end{cases} \\
&\iff \lambda = z_{0,0} + z_{t,0}x^t + \sum_{i=1}^{t-1} z_{i,0}\left(x^i + x^{-i}\right) + z_{0,1}My + z_{1,1}Mxy.
\end{aligned}$$

(ii) The case $n \equiv 0 \bmod 4,\ n \neq 0$: We calculate $H^n(Q_t, {}_\varphi\Lambda) = \mathrm{Ker}\ \delta_1^{\#}/\mathrm{Im}\ \delta_4^{\#}$. It suffices to show that $\mathrm{Im}\ \delta_4^{\#} = 4t\mathbb{Z} \oplus 4t\mathbb{Z}x^t \oplus \bigoplus_{i=1}^{t-1} 2t\mathbb{Z}\left(x^i + x^{-i}\right) \oplus 4\mathbb{Z}My \oplus 4\mathbb{Z}Mxy$. For $\lambda = \sum_{i=0}^{2t-1}\sum_{j=0}^{1} z_{i,j}x^iy^j \in \Lambda$, we have

$$\begin{aligned}
\delta_4^{\#}(\lambda) = N\lambda = 4tz_{0,0} + 4tz_{t,0}x^t &+ \sum_{i=1}^{t-1} 2t\left(z_{i,0} + z_{2t-i,0}\right)\left(x^i + x^{-i}\right) \\
&+ \sum_{i=0}^{t-1} 4z_{2i,1}My + \sum_{i=0}^{t-1} 4z_{2i+1,1}Mxy.
\end{aligned}$$

(iii) The case $n \equiv 1 \bmod 4$: We already know $H^n(Q_t, {}_\varphi\Lambda) = 0$ in Lemma 1.

(iv) The case $n \equiv 2 \bmod 4$ and t even: We calculate $H^n(Q_t, {}_\varphi\Lambda) = \mathrm{Ker}\ \delta_3^{\#}/\mathrm{Im}\ \delta_2^{\#}$. First we show

$$\mathrm{Ker}\ \delta_3^{\#} = \mathbb{Z}(1,0) \oplus \mathbb{Z}(0,1) \oplus \mathbb{Z}(x^t,0) \oplus \mathbb{Z}(0,x^t) \oplus \bigoplus_{i=1}^{t-1} \mathbb{Z}\left(x^i, 0\right)$$

$$\oplus \bigoplus_{i=1}^{t-1} \mathbb{Z}\left(x^i + x^{-i}, 0\right) \oplus \bigoplus_{i=0}^{t-1} \mathbb{Z}\left(tx^{-i}, x^i + x^{-i}\right) \oplus \bigoplus_{i=0}^{2t+1} \mathbb{Z}\boldsymbol{a}_i. \tag{2.2}$$

For $\lambda_1 = \sum_{i=0}^{2t-1}\sum_{j=0}^{1} z_{i,j}x^iy^j$ and $\lambda_2 = \sum_{i=0}^{2t-1}\sum_{j=0}^{1} w_{i,j}x^iy^j \in \Lambda$ $(z_{i,j}, w_{i,j} \in \mathbb{Z})$, we have

$$\begin{aligned}
&(\lambda_1, \lambda_2) \in \operatorname{Ker} \delta_3^{\#} \\
\Longleftrightarrow\ &(x-1)\lambda_1 - (xy-1)\lambda_2 = 0 \\
\Longleftrightarrow\ &\sum_{i=1}^{2t-1} \left(w_{i,0} - w_{2t-i,0}\right) x^i + \left(z_{2t-2,1} - z_{0,1} - w_{2,1} + w_{0,1}\right) y \\
&+ \left(z_{2t-1,1} - z_{1,1}\right) xy + \left(z_{0,1} - z_{2,1} - w_{0,1} + w_{2,1}\right) x^2y \\
&+ \sum_{i=3}^{2t-1} \left(z_{i-2,1} - z_{i,1} - w_{2t+2-i,1} + w_{i,1}\right) x^iy = 0 \\
\Longleftrightarrow\ &\begin{cases}
w_{i,0} - w_{2t-i,0} = 0 \quad (i = 1, 2, \ldots, t-1) \\
z_{2t-2,1} - z_{0,1} - w_{2,1} + w_{0,1} = 0 \\
z_{2t-1,1} - z_{1,1} = 0 \\
z_{0,1} - z_{2,1} - w_{0,1} + w_{2,1} = 0 \\
z_{i-2,1} - z_{i,1} - w_{2t+2-i,1} + w_{i,1} = 0 \quad (i = 3, 4, \ldots, 2t-1)
\end{cases} \\
\Longleftrightarrow\ &\begin{cases}
w_{i,0} - w_{2t-i,0} = 0 \quad (i = 1, 2, \ldots, t-1) \\
\begin{cases}
z_{0,1} - z_{2t-2,1} - w_{0,1} + w_{2,1} = 0 \\
z_{2,1} - z_{2t-2,1} = 0 \\
z_{2i,1} - z_{2t-2,1} - \sum_{j=2}^{i} w_{2j,1} + \sum_{j=2}^{i} w_{2t+2-2j,1} = 0 \\
\qquad\qquad (i = 2, 3, \ldots, t-2)
\end{cases} \\
\begin{cases}
z_{1,1} - z_{2t-1,1} = 0 \\
z_{2i+1,1} - z_{2t-1,1} - \sum_{j=1}^{i} w_{2j+1,1} + \sum_{j=1}^{i} w_{2t+1-2j,1} = 0 \\
\qquad\qquad (i = 1, 2, \ldots, t-2).
\end{cases}
\end{cases}
\end{aligned}$$

If this is the case, then we have

$$\begin{aligned}
(\lambda_1, \lambda_2) &= \sum_{i=0}^{2t-1}\sum_{j=0}^{1} z_{i,j}\left(x^iy^j, 0\right) + \sum_{i=0}^{2t-1}\sum_{j=0}^{1} w_{i,j}\left(0, x^iy^j\right) \\
&= z_{0,0}(1,0) + w_{0,0}(0,1) + z_{t,0}\left(x^t, 0\right) + w_{t,0}\left(0, x^t\right) \\
&\quad + \sum_{i=1}^{t-1}\left\{z_{i,0}\left(x^i, 0\right) + z_{2t-i,0}\left(x^{-i}, 0\right) + w_{i,0}\left(0, x^i + x^{-i}\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
&+ w_{0,1}\boldsymbol{a}_0 + \sum_{i=1}^{t-2} w_{2i+2,1}\boldsymbol{a}_{2i} - z_{2t-2,1}\boldsymbol{a}_{2t-2} + w_{2,1}\boldsymbol{a}_{2t} \\
&+ \sum_{i=0}^{t-1} w_{2i+1,1}\boldsymbol{a}_{2i+1} + z_{2t-1,1}\boldsymbol{a}_{2t+1}. \\
= {}& z_{0,0}(1,0) + w_{0,0}(0,1) + z_{t,0}\left(x^t,0\right) + w_{t,0}\left(0,x^t\right) \\
&+ \sum_{i=1}^{t-1} \Big\{ \left(z_{i,0} - z_{2t-i,0} + tw_{i,0}\right)\left(x^i,0\right) \\
&\qquad + \left(z_{2t-i,0} - tw_{i,0}\right)\left(x^i + x^{-i},0\right) + w_{i,0}\left(tx^{-i}, x^i + x^{-i}\right) \Big\} \\
&+ w_{0,1}\boldsymbol{a}_0 + \sum_{i=1}^{t-2} w_{2i+2,1}\boldsymbol{a}_{2i} - z_{2t-2,1}\boldsymbol{a}_{2t-2} + w_{2,1}\boldsymbol{a}_{2t} \\
&+ \sum_{i=0}^{t-1} w_{2i+1,1}\boldsymbol{a}_{2i+1} + z_{2t-1,1}\boldsymbol{a}_{2t+1}.
\end{aligned}
$$

Hence (λ_1, λ_2) is an element of the right hand side of (2.2). Conversely, $\delta_3^\#$ sends the $\mathbb{Z}$-basis of the right hand side of (2.2) into 0. Hence the reverse inclusion also holds. Thus (2.2) is proved. From (2.1) and (2.2), we get

$$
\begin{aligned}
\operatorname{Ker}\,\delta_3^\# = {}& \mathbb{Z}(1,0) \oplus \mathbb{Z}(0,1) \oplus \mathbb{Z}(x^t,0) \oplus \mathbb{Z}(0,x^t) \\
&\oplus \bigoplus_{i=1}^{t-1} \mathbb{Z}\left(x^i,0\right) \oplus \bigoplus_{i=1}^{t-1} \mathbb{Z}\left(x^i + x^{-i},0\right) \oplus \bigoplus_{i=0}^{t-1} \mathbb{Z}\left(tx^{-i}, x^i + x^{-i}\right) \\
&\oplus \mathbb{Z}(y,y) \oplus \mathbb{Z}(0,xy) \oplus \bigoplus_{i=1}^{2t-2} \mathbb{Z}\boldsymbol{b}_i \oplus \mathbb{Z}\boldsymbol{b}_{2t} \oplus \mathbb{Z}\boldsymbol{b}_{2t+1}.
\end{aligned} \tag{2.3}
$$

Next, we verify

$$
\begin{aligned}
\operatorname{Im}\,\delta_2^\# = {}& 2\mathbb{Z}(1,0) \oplus 2\mathbb{Z}(0,1) \oplus 2\mathbb{Z}(x^t,0) \oplus 2\mathbb{Z}(0,x^t) \oplus \bigoplus_{i=1}^{t-1} 2t\mathbb{Z}\left(x^i,0\right) \\
&\oplus \bigoplus_{i=1}^{t-1} \mathbb{Z}\left(x^i + x^{-i},0\right) \oplus \bigoplus_{i=0}^{t-1} \mathbb{Z}\left(tx^{-i}, x^i + x^{-i}\right) \\
&\oplus 4\mathbb{Z}(y,y) \oplus 4\mathbb{Z}(0,xy) \oplus \bigoplus_{i=1}^{2t-2} \mathbb{Z}\boldsymbol{b}_i \oplus \mathbb{Z}\boldsymbol{b}_{2t} \oplus \mathbb{Z}\boldsymbol{b}_{2t+1}.
\end{aligned} \tag{2.4}
$$

Since, for $\lambda_1 = \sum_{i=0}^{2t-1} \sum_{j=0}^{1} z_{i,j} x^i y^j$ and $\lambda_2 = \sum_{i=0}^{2t-1} \sum_{j=0}^{1} w_{i,j} x^i y^j \in \Lambda$ $(z_{i,j}, w_{i,j} \in \mathbb{Z})$,

$$\begin{aligned}
L\lambda_1 - (y+1)\lambda_2 &= t \sum_{i=0}^{2t-1} z_{i,0} x^i + \sum_{i=0}^{t-1} z_{2i,1} My + \sum_{i=0}^{t-1} z_{2i+1,1} Mxy \\
&\quad - \sum_{i=0}^{2t-1} w_{i,0} \left(x^i + x^{-i}\right) - \sum_{i=0}^{2t-1} w_{i,1} \left(x^i + x^{-i}\right) y, \\
(xy+1)\lambda_1 + (x-1)\lambda_2 &= \sum_{i=0}^{2t-1} z_{i,0} \left(x^i + x^{-i}\right) + \sum_{i=0}^{2t-1} z_{i,1} \left(x^{2-i} + x^i\right) y \\
&\quad + \sum_{i=0}^{2t-1} w_{i,1} \left(x^{i+2} - x^i\right) y,
\end{aligned}$$

we have

$$\begin{aligned}
&\delta_2^{\#}(\lambda_1, \lambda_2) \\
&= z_{0,0}(t,2) - w_{0,0}(2,0) + z_{t,0}\left(tx^t, 2x^t\right) - w_{t,0}\left(2x^t, 0\right) \\
&\quad + \sum_{i=1}^{t-1} \Big\{ z_{i,0}\left(tx^i, x^i + x^{-i}\right) + z_{2t-i,0}\left(tx^{-i}, x^i + x^{-i}\right) \\
&\qquad\qquad - \left(w_{i,0} + w_{2t-i,0}\right)\left(x^i + x^{-i}, 0\right) \Big\} \\
&\quad + \sum_{i=0}^{2t-1} w_{i,1} \boldsymbol{b}_i + \left(z_{0,1} + z_{2,1}\right) \boldsymbol{b}_{2t} + z_{1,1} \boldsymbol{b}_{2t+1} + \sum_{i=1}^{\frac{t}{2}-1} \left(z_{2i+2,1} + z_{2t-2i,1}\right) \boldsymbol{b}'_{2i} \\
&\quad + \sum_{i=1}^{\frac{t}{2}-1} \left(z_{2i+1,1} + z_{2t+1-2i,1}\right) \boldsymbol{b}'_{2i-1} + z_{t+1,1} \boldsymbol{b}'_{t-1} \\
&= 2\left(z_{0,0} t/2 - w_{0,0}\right)(1,0) + 2 z_{0,0}(0,1) + 2\left(z_{t,0} t/2 - w_{t,0}\right)\left(x^t, 0\right) \\
&\quad + 2 z_{t,0}\left(0, x^t\right) + \sum_{i=1}^{t-1} \Big\{ 2t z_{i,0}\left(x^i, 0\right) + \left(z_{i,0} + z_{2t-i,0}\right)\left(tx^{-i}, x^i + x^{-i}\right) \\
&\qquad\qquad - \left(t z_{i,0} + w_{i,0} + w_{2t-i,0}\right)\left(x^i + x^{-i}, 0\right) \Big\} \\
&\quad - 4 w_{0,1} \boldsymbol{a}_0 + \sum_{i=1}^{t-1} \left(w_{0,1} + w_{2i,1}\right) \boldsymbol{b}_{2i} + \left(z_{0,1} + z_{2,1} + 2 w_{0,1}\right) \boldsymbol{b}_{2t} \\
&\quad + 4 w_{2t-1,1} \boldsymbol{a}_1 + \sum_{i=0}^{t-2} \left(w_{2i+1,1} - w_{2t-1,1}\right) \boldsymbol{b}_{2i+1} + \left(z_{1,1} - 2 w_{2t-1,1}\right) \boldsymbol{b}_{2t+1}
\end{aligned}$$

$$+\sum_{i=1}^{\frac{t}{2}-1}(z_{2i+2,1}+z_{2t-2i,1})\boldsymbol{b}'_{2i}+\sum_{i=1}^{\frac{t}{2}-1}(z_{2i+1,1}+z_{2t+1-2i,1})\boldsymbol{b}'_{2i-1}+z_{t+1,1}\boldsymbol{b}'_{t-1}.$$

By Lemma 2 (iv), (v) and (vi), $\boldsymbol{b}'_i$ $(i=1,2,\dots,t-1)$ are expressed by linear combinations of $4\boldsymbol{a}_1,\boldsymbol{b}_1,\boldsymbol{b}_2,\dots,\boldsymbol{b}_{2t-2},\boldsymbol{b}_{2t},\boldsymbol{b}_{2t+1}$. Hence, $\delta_2^{\#}(\lambda_1,\lambda_2)$ is an element of the right hand side of (2.4). On the other hand, in the above expression, it is easily seen that the coefficients of $2(1,0),2(0,1),2(x^t,0),2(0,x^t)$, $2t(x^i,0),(tx^{-i},x^i+x^{-i}),(x^i+x^{-i},0),4\boldsymbol{a}_0,4\boldsymbol{a}_1,\boldsymbol{b}_j,\boldsymbol{b}_{2t},\boldsymbol{b}_{2t+1}$ for $i=1,2,\dots,t-1$ and $j=1,2,\dots,2t-2$ are able to be any elements of $\mathbb{Z}$ by choosing $w_{0,0},z_{0,0},w_{t,0},z_{t,0},z_{i,0},z_{2t-i,0},w_{i,0},w_{0,1},w_{2t-1,1},w_{j,1},z_{0,1},z_{1,1}$ properly. Thus (2.4) is proved. From (2.3) and (2.4), we have

$$\begin{aligned}H^n(Q_t,{}_{\varphi}\Lambda)=\mathbb{Z}(1,0)/2\oplus\mathbb{Z}(0,1)/2\oplus\mathbb{Z}(x^t,0)/2\oplus\mathbb{Z}(0,x^t)/2\\ \oplus\bigoplus_{i=1}^{t-1}\mathbb{Z}\left(x^i,0\right)/2t\oplus\mathbb{Z}(y,y)/4\oplus\mathbb{Z}(0,xy)/4\end{aligned}$$

$$\text{for } n\equiv 2 \mod 4 \text{ and } t \text{ even.}$$

(v) The case $n\equiv 2$ mod 4 and t odd: The calculations are similar to the case t even. We calculate $H^n(Q_t,{}_{\varphi}\Lambda)=\mathrm{Ker}\ \delta_3^{\#}/\mathrm{Im}\ \delta_2^{\#}$. First we show

$$\begin{aligned}\mathrm{Ker}\ \delta_3^{\#}=\mathbb{Z}\left(\frac{t-1}{2},1\right)\oplus\mathbb{Z}(t-2,2)\oplus\mathbb{Z}\left(\frac{t-1}{2}x^t,x^t\right)\oplus\mathbb{Z}((t-2)x^t,2x^t)\\ \oplus\bigoplus_{i=1}^{t-1}\mathbb{Z}\left(x^i,0\right)\oplus\bigoplus_{i=1}^{t-1}\mathbb{Z}\left(x^i+x^{-i},0\right)\oplus\bigoplus_{i=0}^{t-1}\mathbb{Z}\left(tx^{-i},x^i+x^{-i}\right)\\ \oplus\bigoplus_{i=0}^{2t+1}\mathbb{Z}\boldsymbol{a}_i.\end{aligned}\tag{2.5}$$

For $\lambda_1=\sum_{i=0}^{2t-1}\sum_{j=0}^{1}z_{i,j}x^iy^j$ and $\lambda_2=\sum_{i=0}^{2t-1}\sum_{j=0}^{1}w_{i,j}x^iy^j\in\Lambda$ $(z_{i,j},w_{i,j}\in\mathbb{Z})$, we have

$$(\lambda_1,\lambda_2)\in\mathrm{Ker}\ \delta_3^{\#}$$

$$\iff \left\{ \begin{array}{l} w_{i,0} - w_{2t-i,0} = 0 \quad (i = 1, 2, \dots, t-1) \\ \left\{ \begin{array}{l} z_{0,1} - z_{2t-2,1} - w_{0,1} + w_{2,1} = 0 \\ z_{2,1} - z_{2t-2,1} = 0 \\ z_{2i,1} - z_{2t-2,1} - \sum_{j=2}^{i} w_{2j,1} + \sum_{j=2}^{i} w_{2t+2-2j,1} = 0 \\ \qquad\qquad\qquad\qquad\qquad (i = 2, 3, \dots, t-2) \end{array} \right. \\ \left\{ \begin{array}{l} z_{1,1} - z_{2t-1,1} = 0 \\ z_{2i+1,1} - z_{2t-1,1} - \sum_{j=1}^{i} w_{2j+1,1} + \sum_{j=1}^{i} w_{2t+1-2j,1} = 0 \\ \qquad\qquad\qquad\qquad\qquad (i = 1, 2, \dots, t-2). \end{array} \right. \end{array} \right.$$

If this is the case, then we have

$$\begin{aligned} (\lambda_1, \lambda_2) &= (2z_{0,0} - (t-2)w_{0,0}) \left(\frac{t-1}{2}, 1 \right) + (-z_{0,0} + w_{0,0}(t-1)/2)\,(t-2, 2) \\ &\quad + (2z_{t,0} - (t-2)w_{t,0}) \left(\frac{t-1}{2} x^t, x^t \right) \\ &\quad + (-z_{t,0} + w_{t,0}(t-1)/2)\left((t-2)x^t, 2x^t\right) \\ &\quad + \sum_{i=1}^{t-1} \Big\{ (z_{i,0} - z_{2t-i,0} + t w_{i,0}) \left(x^i, 0\right) \\ &\qquad\quad + (z_{2t-i,0} - t w_{i,0}) \left(x^i + x^{-i}, 0\right) + w_{i,0} \left(t x^{-i}, x^i + x^{-i}\right) \Big\} \\ &\quad + w_{0,1}\boldsymbol{a}_0 + \sum_{i=1}^{t-2} w_{2i+2,1}\boldsymbol{a}_{2i} - z_{2t-2,1}\boldsymbol{a}_{2t-2} + w_{2,1}\boldsymbol{a}_{2t} \\ &\quad + \sum_{i=0}^{t-1} w_{2i+1,1}\boldsymbol{a}_{2i+1} + z_{2t-1,1}\boldsymbol{a}_{2t+1}. \end{aligned}$$

Hence (λ_1, λ_2) is an element of the right hand side of (2.5). Conversely, $\delta_3^{\#}$ sends the $\mathbb{Z}$-basis of the right hand side of (2.5) into 0. Hence the reverse inclusion also holds. Thus (2.5) is proved. From (2.1) and (2.5), we get

$$\begin{aligned} \operatorname{Ker} \delta_3^{\#} &= \mathbb{Z} \left(\frac{t-1}{2}, 1 \right) \oplus \mathbb{Z}(t-2, 2) \oplus \mathbb{Z} \left(\frac{t-1}{2} x^t, x^t \right) \oplus \mathbb{Z}((t-2)x^t, 2x^t) \\ &\quad \oplus \bigoplus_{i=1}^{t-1} \mathbb{Z}\left(x^i, 0\right) \oplus \bigoplus_{i=1}^{t-1} \mathbb{Z}\left(x^i + x^{-i}, 0\right) \oplus \bigoplus_{i=0}^{t-1} \mathbb{Z}\left(t x^{-i}, x^i + x^{-i}\right) \\ &\quad \oplus \mathbb{Z}(y, y) \oplus \mathbb{Z}(0, xy) \oplus \bigoplus_{i=1}^{2t-2} \mathbb{Z}\boldsymbol{b}_i \oplus \mathbb{Z}\boldsymbol{b}_{2t} \oplus \mathbb{Z}\boldsymbol{b}_{2t+1}. \end{aligned} \tag{2.6}$$

Next, we verify

$$\begin{aligned}\mathrm{Im}\ \delta_2^{\#} &= 4\mathbb{Z}\left(\frac{t-1}{2},1\right) \oplus \mathbb{Z}(t-2,2) \oplus 4\mathbb{Z}\left(\frac{t-1}{2}x^t, x^t\right) \oplus \mathbb{Z}\left((t-2)x^t, 2x^t\right) \\ &\oplus \bigoplus_{i=1}^{t-1} 2t\mathbb{Z}\left(x^i,0\right) \oplus \bigoplus_{i=1}^{t-1} \mathbb{Z}\left(x^i+x^{-i},0\right) \oplus \bigoplus_{i=0}^{t-1} \mathbb{Z}\left(tx^{-i}, x^i+x^{-i}\right) \\ &\oplus 4\mathbb{Z}(y,y) \oplus 4\mathbb{Z}(0,xy) \oplus \bigoplus_{i=1}^{2t-2} \mathbb{Z}\boldsymbol{b}_i \oplus \mathbb{Z}\boldsymbol{b}_{2t} \oplus \mathbb{Z}\boldsymbol{b}_{2t+1}. \end{aligned} \tag{2.7}$$

For $\lambda_1 = \sum_{i=0}^{2t-1}\sum_{j=0}^{1} z_{i,j}x^iy^j$ and $\lambda_2 = \sum_{i=0}^{2t-1}\sum_{j=0}^{1} w_{i,j}x^iy^j \in \Lambda$ $(z_{i,j}, w_{i,j} \in \mathbb{Z})$, we have

$$\begin{aligned}\delta_2^{\#}(\lambda_1,\lambda_2) &= 4\left(z_{0,0}-w_{0,0}\right)\left(\frac{t-1}{2},1\right) + \left(2w_{0,0}-z_{0,0}\right)(t-2,2) \\ &+ 4\left(z_{t,0}-w_{t,0}\right)\left(\frac{t-1}{2}x^t,x^t\right) + \left(2w_{t,0}-z_{t,0}\right)\left((t-2)x^t,2x^t\right) \\ &+ \sum_{i=1}^{t-1}\left\{2tz_{i,0}\left(x^i,0\right) + \left(z_{i,0}+z_{2t-i,0}\right)\left(tx^{-i},x^i+x^{-i}\right)\right. \\ &\qquad\left. - \left(tz_{i,0}+w_{i,0}+w_{2t-i,0}\right)\left(x^i+x^{-i},0\right)\right\} \\ &- 4w_{0,1}\boldsymbol{a}_0 + \sum_{i=1}^{t-1}\left(w_{0,1}+w_{2i,1}\right)\boldsymbol{b}_{2i} + \left(z_{0,1}+z_{2,1}+2w_{0,1}\right)\boldsymbol{b}_{2t} \\ &+ 4w_{2t-1,1}\boldsymbol{a}_1 + \sum_{i=0}^{t-2}\left(w_{2i+1,1}-w_{2t-1,1}\right)\boldsymbol{b}_{2i+1} \\ &+ \left(z_{1,1}-2w_{2t-1,1}\right)\boldsymbol{b}_{2t+1} + \sum_{i=1}^{\frac{t-3}{2}}\left(z_{2i+2,1}+z_{2t-2i,1}\right)\boldsymbol{b}'_{2i} \\ &+ z_{t+1,1}\boldsymbol{b}'_{t-1} + \sum_{i=1}^{\frac{t-1}{2}}\left(z_{2i+1,1}+z_{2t+1-2i,1}\right)\boldsymbol{b}'_{2i-1}.\end{aligned}$$

By Lemma 2 (iv), (v) and (vi), $\boldsymbol{b}'_i$ $(i=1,2,\ldots,t-1)$ are expressed by linear combinations of $4\boldsymbol{a}_1, \boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_{2t-2}, \boldsymbol{b}_{2t}, \boldsymbol{b}_{2t+1}$. Hence, $\delta_2^{\#}(\lambda_1,\lambda_2)$ is an element of the right hand side of (2.7). On the other hand, in the above expression, it is easily seen that coefficients of $4((t-1)/2,1), (t-2,2), 4((t-1)x^t/2, x^t), ((t-2)x^t,2x^t), 2t(x^i,0), (tx^{-i},x^i+x^{-i}), (x^i+x^{-i},0), 4\boldsymbol{a}_0, 4\boldsymbol{a}_1, \boldsymbol{b}_j, \boldsymbol{b}_{2t}, \boldsymbol{b}_{2t+1}$ for $i=1,2,\ldots,t-1$ and $j=1,2,\ldots,2t-2$ are able to be any elements of $\mathbb{Z}$ by choosing $w_{0,0}, z_{0,0}, w_{t,0}, z_{t,0}, z_{i,0}, z_{2t-i,0}, w_{i,0}, w_{0,1}, w_{2t-1,1}, w_{j,1}, z_{0,1}, z_{1,1}$ properly. Thus (2.7) is proved. From (2.6) and (2.7), we have

$$\begin{aligned} H^n(Q_t, {}_\varphi\Lambda) = \mathbb{Z}\left(\frac{t-1}{2}, 1\right)/4 \oplus \mathbb{Z}\left(\frac{t-1}{2}x^t, x^t\right)/4 \oplus \bigoplus_{i=1}^{t-1} \mathbb{Z}\left(x^i, 0\right)/2t \\ \oplus \mathbb{Z}(y, y)/4 \oplus \mathbb{Z}(0, xy)/4 \quad \text{for } n \equiv 2 \mod 4 \text{ and } t \text{ odd.} \end{aligned}$$

(vi) The case $n \equiv 3$ mod 4: We already know $H^n(Q_t, {}_\varphi\Lambda) = 0$ in Lemma 1. This completes the proof of Proposition 1. □

§3. Ring structure of $H^*(Q_t, {}_\varphi\Lambda)$

We maintain the notation of Section 2: let $\Lambda = \mathbb{Z}Q_t$, where

$$Q_t = \langle x, y | x^{2t} = 1, x^t = y^2, yxy^{-1} = x^{-1} \rangle,$$

the generalized quaternion group of order $4t$ for any positive integer $t \geqslant 2$. In this section, we will determine the ring structure of $H^*(Q_t, {}_\varphi\Lambda)$ with the product $\smile$ given by the composition of the following maps as in Section 1:

$$H^p(Q_t, {}_\varphi\Lambda) \otimes H^q(Q_t, {}_\varphi\Lambda) \xrightarrow{\smile_\mu} H^{p+q}(Q_t, {}_\varphi(\Lambda \otimes_\Lambda \Lambda)) \xrightarrow{\sim} H^{p+q}(Q_t, {}_\varphi\Lambda).$$

In Section 3.1, we will calculate the cup product for the generators of $H^*(Q_t, {}_\varphi\Lambda)$ for t even, and for t odd in Section 3.2. In Section 3.3, we give an explicit description of the cohomology ring $H^*(Q_t, {}_\varphi\Lambda)$ summarizing the calculations in Sections 3.1 and 3.2.

To compute the cup product, we use a diagonal approximation Δ_Y on the periodic resolution (Y, δ) of period 4. A construction of $(\Delta_Y)_{p,q}$ follows from [HaSa, Sections 1 and 2]; let (X, d) be the standard resolution of Q_t and $\Delta_{p,q}$ a diagonal approximation on (X, d). We define $(\Delta_Y)_{p,q}$ on (Y, δ) by

$$(\Delta_Y)_{p,q} := u_p \otimes u_q \cdot \Delta_{p,q} \cdot v_{p+q}, \tag{3.1}$$

where u_i denotes a chain transformation $X_i \to Y_i$ and v_j denotes a chain transformation $Y_j \to X_j$. Then it is easy to see that the following equations hold:

$$\begin{aligned} &(\Delta_Y)_{p,q}\, \delta_{p+q+1} = (\delta_{p+1} \otimes 1_q)\,(\Delta_Y)_{p+1,q} + (-1)^p\,(1_p \otimes \delta_{q+1})\,(\Delta_Y)_{p,q+1}\,, \\ &(\varepsilon \otimes \varepsilon)\,(\Delta_Y)_{0,0} = \varepsilon. \end{aligned}$$

Therefore $(\Delta_Y)_{p,q}$ is a diagonal approximation on (Y, δ). Next we calculate $(\Delta_Y)_{0,2}$ and $(\Delta_Y)_{2,2}$ using $\Delta_{p,q}$ and the explicit formulas of u_i and v_j which are given in [HaSa, Propositions 1 and 2].

Lemma 3. *The above $(\mathit{\Delta}_Y)_{p,q}$ satisfies the following equations for $p = 0, 2$ and $q = 2$:*

$$
\begin{aligned}
&(\mathit{\Delta}_Y)_{0,2}(1,0) = 1 \otimes (1,0), \quad (\mathit{\Delta}_Y)_{0,2}(0,1) = 1 \otimes (0,1);\\
&(\mathit{\Delta}_Y)_{2,2}(1) = -\big(1 - x^t y, Lxy\big) \otimes (-1,0) + \big(1 - x^t y, Lxy\big) \otimes x^{-1}y(-1,0)\\
&\qquad - \sum_{i=0}^{2t-1} x^{i-1}(0,1) \otimes x^{i-1}y(-1,0) - \big(1 - x^t y, Lxy\big) \otimes y(0,1)\\
&\qquad - \sum_{i=0}^{2t-1} x^{i-1}(-1,1) \otimes x^{i+t-1}(0,1)\\
&\qquad - \sum_{i=t+1}^{2t-1} \big(1 - x^t y, Lxy\big) \otimes x^{i+t-1}(0,1).
\end{aligned}
$$

Proof. In the case $p = 0$ and $q = 2$, we have the following:

$$
\begin{aligned}
(\mathit{\Delta}_Y)_{0,2}(1,0) &= u_0 \otimes u_2 \cdot \mathit{\Delta}_{0,2} \cdot v_2(1,0)\\
&= u_0 \otimes u_2 \cdot \mathit{\Delta}_{0,2} \left(\sum_{i=1}^{t-1} [x^i|x] - [y|y] \right)\\
&= u_0 \otimes u_2 \left(\sum_{i=1}^{t-1} [\cdot] \otimes [x^i|x] - [\cdot] \otimes [y|y] \right)\\
&= 1 \otimes (1,0),\\
(\mathit{\Delta}_Y)_{0,2}(0,1) &= u_0 \otimes u_2 \cdot \mathit{\Delta}_{0,2} \cdot v_2(0,1)\\
&= u_0 \otimes u_2 \left([\cdot] \otimes [x|y] + [\cdot] \otimes [xy|x] \right)\\
&= 1 \otimes (0,1).
\end{aligned}
$$

In the above, $\sigma_0[\cdot]$ denotes $\sigma_0 \in X_0$ and $\sigma_0[\sigma_1|\sigma_2]$ denotes $\sigma_0 \otimes \sigma_1 \otimes \sigma_2 \in X_2$ for $\sigma_i \in Q_t$. In the case $p = q = 2$, the explicit formula of $(\mathit{\Delta}_Y)_{2,2}$ is given in [HaSa, Section 4]. Thus the proof is complete. $\square$

3.1. The case t even.

In this subsection, we consider the case t even. First of all, we calculate the products of the generators of $H^0(Q_t, {}_\varphi \mathit{\Lambda})$. By Proposition 1, we take the generators as follows:

$$
\begin{aligned}
A_0 = 1,\ B_0 &= x^t,\\
(C_i)_0 &= x^i + x^{-i} \quad (i = 1, 2, \dots, t-1),
\end{aligned}
$$

$$D_0 = My,\ E_0 = Mxy.$$

Note that A_0 is the identity element of the cohomology ring $H^*(Q_t, {}_\varphi\Lambda)$.

Proposition 2. *The following equations hold in $H^0(Q_t, {}_\varphi\Lambda)$ for the generators $A_0, B_0, (C_i)_0$ $(i = 1, 2, \dots, t-1), D_0, E_0$ of $H^0(Q_t, {}_\varphi\Lambda)$:*

$$B_0{}^2 = A_0,\ B_0(C_i)_0 = (C_{t-i})_0,\ B_0D_0 = D_0,\ B_0E_0 = E_0,$$
$$(C_i)_0D_0 = \begin{cases} 2D_0 & (i\ \textit{even}) \\ 2E_0 & (i\ \textit{odd}), \end{cases} \quad (C_i)_0E_0 = \begin{cases} 2E_0 & (i\ \textit{even}) \\ 2D_0 & (i\ \textit{odd}), \end{cases}$$
$$D_0{}^2 = E_0{}^2 = tA_0 + tB_0 + t\sum_{l=1}^{\frac{t}{2}-1}(C_{2l})_0,\ D_0E_0 = t\sum_{l=0}^{\frac{t}{2}-1}(C_{2l+1})_0,$$
$$(C_i)_0(C_j)_0 = U_{i+j} + U_{i-j},$$
$$\textit{where } U_k = \begin{cases} (C_{-k})_0 & (-t < k < 0) \\ 2A_0 & (k = 0) \\ (C_k)_0 & (0 < k < t) \\ 2B_0 & (k = t) \\ (C_{2t-k})_0 & (t < k < 2t). \end{cases}$$

Proof. Note that the relations of degree 0 correspond to the multiplication in $Z\Lambda$. So we obtain, for example,

$$B_0(C_i)_0 = x^t\left(x^i + x^{-i}\right) = x^{-t+i} + x^{t-i} = (C_{t-i})_0 \qquad (i = 1, 2, \dots, t-1).$$

The other calculations are done similarly. □

By Proposition 1, we take the generators of $H^2(Q_t, {}_\varphi\Lambda)$ as follows:

$$(A_\alpha)_2 = (1, 0),\ (A_\beta)_2 = (0, 1),$$
$$(B_\alpha)_2 = (x^t, 0),\ (B_\beta)_2 = (0, x^t),$$
$$(C_i)_2 = \left(x^i, 0\right) \quad (i = 1, 2, \dots, t-1),$$
$$D_2 = (y, y),\ E_2 = (0, xy).$$

Next, we compute the cup product in degree 2. The following Lemma 4 is useful for the calculation.

Lemma 4. *For $i = 1, 2, \dots, t-1$, the following particular equations hold in $H^2(Q_t, {}_\varphi\Lambda)$:*

$$\text{(i)} \quad \left(x^{t+i}, 0\right) = -\left(C_{t-i}\right)_2,$$

$$\begin{aligned}
&\text{(ii)} && \left(0, x^i + x^{-i}\right) = t\,(C_i)_2\,, \\
&\text{(iii)} && \left(x^t y, x^t y\right) = -D_2, \\
&\text{(iv)} && \left(0, x^{t+1} y\right) = -E_2, \\
&\text{(v)} && \left(Mx^j y, 0\right) = \begin{cases} 2D_2 & (j\ \mathit{even}) \\ 2E_2 & (j\ \mathit{odd}), \end{cases} \\
&\text{(vi)} && \left((x^i + x^{-i})y, (x^i + x^{-i})y\right) = \begin{cases} 2D_2 & (i\ \mathit{even}) \\ 2E_2 & (i\ \mathit{odd}), \end{cases} \\
&\text{(vii)} && \left(0, Mx^j y\right) = \begin{cases} tD_2 & (j\ \mathit{even}) \\ (2-t)E_2 & (j\ \mathit{odd}), \end{cases} \\
&\text{(viii)} && \left(0, (x^i + x^{-i})xy\right) = \begin{cases} 2E_2 & (i\ \mathit{even}) \\ 2D_2 & (i\ \mathit{odd}). \end{cases}
\end{aligned}$$

Proof. In this proof, we use the notation introduced in Section 2.1.
(i), (ii): By (2.4), $\left(x^k + x^{-k}, 0\right) = \left(tx^{-k}, x^k + x^{-k}\right) = 0$ hold in $H^2(Q_{t,\varphi}\Lambda)$ for $k = 1, 2, \ldots, t-1$. Hence the following equations hold in $H^2(Q_{t,\varphi}\Lambda)$:

$$\begin{aligned}
\left(x^{t+i}, 0\right) &= \left(x^{t-i} + x^{i-t}, 0\right) - \left(x^{t-i}, 0\right) = -\,(C_{t-i})_2\,, \\
\left(0, x^i + x^{-i}\right) &= t(x^i, 0) - t\left(x^i + x^{-i}, 0\right) + \left(tx^{-i}, x^i + x^{-i}\right) = t\,(C_i)_2\,.
\end{aligned}$$

(iii), (iv) and (v): By Lemma 2 (i), (ii) and (2.4), the following equations hold in $H^2(Q_{t,\varphi}\Lambda)$:

$$\begin{aligned}
\left(x^t y, x^t y\right) &= \boldsymbol{a}_{t-2} = -\boldsymbol{a}_0 + \sum_{k=1}^{\frac{t}{2}-1} \boldsymbol{b}_{2k} + \boldsymbol{b}_{2t} = -(y, y) = -D_2, \\
\left(0, x^{t+1} y\right) &= \boldsymbol{a}_{t+1} = -\boldsymbol{a}_1 + \sum_{k=0}^{\frac{t}{2}-1} \boldsymbol{b}_{2k+1} + \boldsymbol{b}_{2t+1} = -(0, xy) = -E_2, \\
(My, 0) &= -\boldsymbol{a}_{2t-2} = 2\boldsymbol{a}_0 - \sum_{k=1}^{t-1} \boldsymbol{b}_{2k} - \boldsymbol{b}_{2t} = 2(y, y) = 2D_2, \\
(Mxy, 0) &= \boldsymbol{a}_{2t+1} = -2\boldsymbol{a}_1 + \boldsymbol{b}_{2t+1} = 2(0, xy) = 2E_2.
\end{aligned}$$

(vi), (vii) and (viii): Note that the following equations hold:

$$\left((x^i + x^{-i})y, (x^i + x^{-i})y\right) = \begin{cases} 2\boldsymbol{a}_1 - \boldsymbol{b}_{2t-1} & (i = 1) \\ \boldsymbol{a}_{2t-2} - \boldsymbol{b}_{2t-2} + \boldsymbol{b}_{2t} & (i = 2) \\ 2\boldsymbol{a}_1 - \boldsymbol{b}_{2t-i} - \boldsymbol{b}_{2t+1} + \boldsymbol{b}'_{i-2} & (i\ \text{odd},\ i \geqslant 3) \\ \boldsymbol{a}_{2t-2} - \boldsymbol{b}_{2t-i} + \boldsymbol{b}'_{i-2} & (i\ \text{even},\ i \geqslant 4), \end{cases}$$

$$(0, My) = (t/2)\boldsymbol{a}_{2t-2} + \boldsymbol{b}_{2t} + \sum_{k=1}^{\frac{t}{2}-1} \boldsymbol{b}'_{2k},$$

$$(0, Mxy) = \boldsymbol{a}_1 + \boldsymbol{a}_{t+1} + (1 - t/2)\boldsymbol{a}_{2t+1} + \sum_{k=1}^{\frac{t}{2}-1} \boldsymbol{b}'_{2k-1},$$

$$\big(0, (x^i + x^{-i})xy\big) = \begin{cases} \boldsymbol{a}_{2t-2} + \boldsymbol{b}_{2t} & (i = 1) \\ 2\boldsymbol{a}_1 - \boldsymbol{b}_{2t+1} + \boldsymbol{b}'_{i-1} & (i \text{ even}) \\ \boldsymbol{a}_{2t-2} + \boldsymbol{b}'_{i-1} & (i \text{ odd},\ i \geqslant 3). \end{cases}$$

By Lemma 2 (iv), (v) and (vi), $\boldsymbol{b}'_i$ $(i = 1, 2, \dots, t-1)$ are expressed by linear combinations of $4\boldsymbol{a}_1, \boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_{2t-2}, \boldsymbol{b}_{2t}, \boldsymbol{b}_{2t+1}$. Applying (2.4), (iv) and (v) of this lemma to the above equations, we have the results. This completes the proof. □

Proposition 3. *The following equations hold in $H^2(Q_t, {}_\varphi\Lambda)$ for the generators $A_0, B_0, (C_i)_0$ $(i = 1, 2, \dots, t-1), D_0, E_0$ of $H^0(Q_t, {}_\varphi\Lambda)$ and the generators $(A_\alpha)_2, (A_\beta)_2, (B_\alpha)_2, (B_\beta)_2, (C_i)_2$ $(i = 1, 2, \dots, t-1), D_2, E_2$ of $H^2(Q_t, {}_\varphi\Lambda)$:*

(i) $2(A_\alpha)_2 = 2(A_\beta)_2 = 2(B_\alpha)_2 = 2(B_\beta)_2 = 2t(C_i)_2 = 4D_2 = 4E_2 = 0.$

(ii) $B_0(A_\alpha)_2 = (B_\alpha)_2,\ B_0(A_\beta)_2 = (B_\beta)_2,\ B_0(B_\alpha)_2 = (A_\alpha)_2,$
$B_0(B_\beta)_2 = (A_\beta)_2,\ B_0(C_i)_2 = -(C_{t-i})_2,\ B_0D_2 = -D_2,\ B_0E_2 = -E_2.$

(iii) $(C_i)_0(A_\alpha)_2 = (C_i)_0(B_\alpha)_2 = 0,\ (C_i)_0(A_\beta)_2 = t(C_i)_2,$
$(C_i)_0(B_\beta)_2 = t(C_{t-i})_2,$

$$(C_i)_0D_2 = \begin{cases} 2D_2 & (i \textit{ even}) \\ 2E_2 & (i \textit{ odd}), \end{cases} \quad (C_i)_0E_2 = \begin{cases} 2E_2 & (i \textit{ even}) \\ 2D_2 & (i \textit{ odd}), \end{cases}$$

$(C_i)_0(C_j)_2 = V_{i+j} + V_{j-i},$

$$\textit{where } V_k = \begin{cases} -(C_{-k})_2 & (-t < k < 0) \\ (A_\alpha)_2 & (k = 0) \\ (C_k)_2 & (0 < k < t) \\ (B_\alpha)_2 & (k = t) \\ -(C_{2t-k})_2 & (t < k < 2t). \end{cases}$$

(iv) $D_0(A_\alpha)_2 = D_0(B_\alpha)_2 = 2D_2,$

$$D_0(A_\beta)_2 = D_0(B_\beta)_2 = \begin{cases} 0 & (t \equiv 0 \mod 4) \\ 2D_2 & (t \equiv 2 \mod 4), \end{cases}$$

$$D_0(C_i)_2 = \begin{cases} 2D_2 & (i \textit{ even}) \\ 2E_2 & (i \textit{ odd}), \end{cases}$$

$$D_0D_2 = (A_\alpha)_2 + (A_\beta)_2 + (B_\alpha)_2 + (B_\beta)_2 + t\sum_{l=1}^{\frac{t}{2}-1}(C_{2l})_2,$$

$$D_0E_2 = t\sum_{l=0}^{\frac{t}{2}-1}(C_{2l+1})_2.$$

(v) $E_0\,(A_\alpha)_2 = E_0\,(B_\alpha)_2 = 2E_2,$

$$E_0\,(A_\beta)_2 = E_0\,(B_\beta)_2 = \begin{cases} 2E_2 & (t \equiv 0 \mod 4) \\ 0 & (t \equiv 2 \mod 4), \end{cases}$$

$$E_0(C_i)_2 = \begin{cases} 2E_2 & (i\ \textit{even}) \\ 2D_2 & (i\ \textit{odd}), \end{cases} \quad E_0D_2 = t\sum_{l=0}^{\frac{t}{2}-1}(C_{2l+1})_2,$$

$$E_0E_2 = (A_\beta)_2 + (B_\beta)_2 + t\sum_{l=1}^{\frac{t}{2}-1}(C_{2l})_2.$$

Proof. See Proposition 1 for (i). The other relations in degree 2 are obtained by means of the following homomorphisms:

$$\begin{aligned} \Lambda \otimes \Lambda^2 &\xrightarrow{\alpha_0^{-1}\otimes\alpha_2^{-1}} \mathrm{Hom}_\Lambda(Y_0, {}_\varphi\Lambda) \otimes \mathrm{Hom}_\Lambda(Y_2, {}_\varphi\Lambda) \\ &\xrightarrow{(\Delta_Y)_{0,2}^{\#}} \mathrm{Hom}_\Lambda\,(Y_2, {}_\varphi\Lambda \otimes {}_\varphi\Lambda) \\ &\xrightarrow{\mathrm{Hom}(\mathrm{id}_{Y_2},\mu)} \mathrm{Hom}_\Lambda\,(Y_2, {}_\varphi\Lambda) \\ &\xrightarrow{\alpha_2} \Lambda^2, \end{aligned}$$

where α_2 denotes the isomorphism $\mathrm{Hom}_\Lambda\,(Y_2, {}_\varphi\Lambda) \xrightarrow{\sim} \Lambda^2$ stated in Section 2, and so on. Let S_0 denote any generator s_0 of $H^0(Q_t, {}_\varphi\Lambda)$ as a $\mathbb{Z}$-module and S_2 denote any generator (s_2, s_2') of $H^2(Q_t, {}_\varphi\Lambda)$ as a $\mathbb{Z}$-module. Since

$$\begin{aligned} \left(\alpha_0^{-1}(S_0) \otimes \alpha_2^{-1}(S_2)\right)\Big((\Delta_Y)_{0,2}\,(1,0)\Big) &= s_0 \otimes s_2, \\ \left(\alpha_0^{-1}(S_0) \otimes \alpha_2^{-1}(S_2)\right)\Big((\Delta_Y)_{0,2}\,(0,1)\Big) &= s_0 \otimes s_2', \end{aligned}$$

it follows $S_0S_2 = (s_0s_2, s_0s_2')$ holds in $H^2(Q_t, {}_\varphi\Lambda)$.
(ii) The products of B_0 and the generators of $H^2(Q_t, {}_\varphi\Lambda)$: By the above, we have the following:

$$\begin{aligned} B_0\,(A_\alpha)_2 &= (x^t, 0) = (B_\alpha)_2\,, \\ B_0\,(A_\beta)_2 &= (0, x^t) = (B_\beta)_2\,, \end{aligned}$$

$$
\begin{aligned}
B_0\,(B_\alpha)_2 &= (1,0) = (A_\alpha)_2\,,\\
B_0\,(B_\beta)_2 &= (0,1) = (A_\beta)_2\,,\\
B_0\,(C_i)_2 &= \left(x^{t+i},0\right) = -\,(C_{t-i})_2 \quad \text{(by Lemma 4 (i))},\\
B_0 D_2 &= (x^t y, x^t y) = -D_2 \quad \text{(by Lemma 4 (iii))},\\
B_0 E_2 &= (0, x^{t+1} y) = -E_2 \quad \text{(by Lemma 4 (iv))}.
\end{aligned}
$$

(iii) The products of $(C_i)_0$ and the generators of $H^2(Q_t, {}_\varphi\Lambda)$: By the similar calculation as above, we have the following:

$$
\begin{aligned}
&(C_i)_0\,(A_\alpha)_2 = (C_i)_0\,(B_\alpha)_2 = 0 \quad \text{(by (2.4))},\\
&(C_i)_0\,(A_\beta)_2 = (0, x^i + x^{-i}) = t\,(C_i)_2 \quad \text{(by Lemma 4 (ii))},\\
&(C_i)_0\,(B_\beta)_2 = (0, x^{t-i} + x^{-t+i}) = t\,(C_{t-i})_2 \quad \text{(by Lemma 4 (ii))},\\
&\quad (C_i)_0\, D_2 = \left((x^i + x^{-i})y, (x^i + x^{-i})y\right) = \begin{cases} 2D_2 & (i \text{ even}) \\ 2E_2 & (i \text{ odd}) \end{cases}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 4 (iv))},\\
&\quad (C_i)_0\, E_2 = \left(0, (x^i + x^{-i})xy\right) = \begin{cases} 2E_2 & (i \text{ even}) \\ 2D_2 & (i \text{ odd}) \end{cases} \quad \text{(by Lemma 4 (iv))}.
\end{aligned}
$$

Note that the following equation holds in $H^2(Q_t, {}_\varphi\Lambda)$ using Lemma 4 (i):

$$
(x^k, 0) = \begin{cases} (A_\alpha)_2 & (k = 0) \\ (C_k)_2 & (0 < k < t) \\ (B_\alpha)_2 & (k = t) \\ -(C_{2t-k})_2 & (t < k < 2t). \end{cases}
$$

Checking all cases for $i+j$ and $j-i$, we have $(C_i)_0\,(C_j)_2 = (x^{i+j} + x^{j-i}, 0) = V_{i+j} + V_{j-i}$.

(iv) The products of D_0 and the generators of $H^2(Q_t, {}_\varphi\Lambda)$: Similarly we have the following:

$$
\begin{aligned}
&D_0\,(A_\alpha)_2 = D_0\,(B_\alpha)_2 = (My, 0) = 2D_2 \quad \text{(by Lemma 4 (v))},\\
&D_0\,(A_\beta)_2 = D_0\,(B_\beta)_2 = (0, My) = \begin{cases} 0 & (t \equiv 0 \mod 4) \\ 2D_2 & (t \equiv 2 \mod 4) \end{cases}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 4 (vii))},\\
&\ D_0\,(C_i)_2 = (Mx^{-i}y, 0) = \begin{cases} 2D_2 & (i \text{ even}) \\ 2E_2 & (i \text{ odd}) \end{cases} \quad \text{(by Lemma 4 (v))},
\end{aligned}
$$

$$D_0D_2 = (M, M) = (A_\alpha)_2 + (A_\beta)_2 + (B_\alpha)_2 + (B_\beta)_2 + t\sum_{l=1}^{\frac{t}{2}-1}(C_{2l})_2$$
$$\text{(by (2.4) and Lemma 4 (ii))},$$
$$D_0E_2 = (0, Mx) = t\sum_{l=0}^{\frac{t}{2}-1}(C_{2l+1})_2 \qquad \text{(by Lemma 4 (ii))}.$$

(v) The products of E_0 and the generators of $H^2(Q_t, {}_\varphi\Lambda)$: Similarly we have the following:

$$E_0\,(A_\alpha)_2 = E_0\,(B_\alpha)_2 = (Mxy, 0) = 2E_2,$$
$$E_0\,(A_\beta)_2 = E_0\,(B_\beta)_2 = (0, Mxy) = \begin{cases} 2E_2 & (t \equiv 0 \mod 4) \\ 0 & (t \equiv 2 \mod 4), \end{cases}$$

$$E_0\,(C_i)_2 = (Mx^{1-i}y, 0) = \begin{cases} 2E_2 & (i \text{ even}) \\ 2D_2 & (i \text{ odd}), \end{cases}$$
$$E_0D_2 = (Mx, Mx) = t\sum_{l=0}^{\frac{t}{2}-1}(C_{2l+1})_2,$$
$$E_0E_2 = (0, M) = (A_\beta)_2 + (B_\beta)_2 + t\sum_{l=1}^{\frac{t}{2}-1}(C_{2l})_2.$$

This completes the proof of Proposition 3. □

Remark 1. Since $\mathbb{Z}$ is a Q_t-direct summand of $\Lambda = \mathbb{Z}Q_t$ using the embedding map $\mathbb{Z} \to \Lambda$ by $1 \mapsto 1$, we have the following monomorphism of the complete cohomology rings:

$$\hat{H}^*(Q_t, \mathbb{Z}) := \bigoplus_{r\in\mathbb{Z}} \hat{H}^r(Q_t, \mathbb{Z}) \to \hat{H}^*(Q_t, {}_\varphi\Lambda) := \bigoplus_{r\in\mathbb{Z}} \hat{H}^r(Q_t, {}_\varphi\Lambda).$$

Now we put $A_4 = 1$ which is a generator of $H^4(Q_t, {}_\varphi\Lambda)$. By the above map, A_4 is the image of an invertible element of order $4t$ in $H^4(Q_t, \mathbb{Z})$. Hence A_4 is also an invertible element in $\hat{H}^*(Q_t, {}_\varphi\Lambda)$. The cup product with A_4 gives a periodicity isomorphism

$$A_4 \smile - : H^r(Q_t, {}_\varphi\Lambda) \xrightarrow{\sim} H^{r+4}(Q_t, {}_\varphi\Lambda)$$

for all $r \geq 0$. See also [Sa1, Section 3.1].

In particular, if we put $r = 0$, we have the following proposition:

Proposition 4. *The generators of $H^4(Q_t, {}_\varphi\Lambda)$ are expressed by the product of A_4 and the generators $A_0, B_0, (C_i)_0$ $(i = 1, 2, \ldots, t-1), D_0, E_0$ of $H^0(Q_t, {}_\varphi\Lambda)$, that is, the following equations hold in $H^4(Q_t, {}_\varphi\Lambda)$:*

$$x^t = A_4 B_0, \ x^i + x^{-i} = A_4 \left(C_i\right)_0 \quad (i = 1, 2, \ldots, t-1),$$
$$My = A_4 D_0, \ Mxy = A_4 E_0.$$

Proof. We use the notation introduced in the beginning of this section; let (X, d) be the standard resolution of Q_t and $\Delta_{p,q}$ a diagonal approximation on (X, d). We denote cocycles representing generators of $H^4(Q_t, {}_\varphi\Lambda)$ and $H^0(Q_t, {}_\varphi\Lambda)$ by $f \in \mathrm{Hom}_\Lambda(X_4, {}_\varphi\Lambda)$ and $g \in \mathrm{Hom}_\Lambda(X_0, {}_\varphi\Lambda)$, respectively. Then we have

$$(f \smile g)(\sigma_0[\sigma_1|\ldots|\sigma_4]) = f(\sigma_0[\sigma_1|\ldots|\sigma_4])g([\cdot]). \tag{3.2}$$

In the above, $\sigma_0[\cdot]$ denotes $\sigma_0 \in X_0$ and $\sigma_0[\sigma_1|\ldots|\sigma_4]$ denotes $\sigma_0 \otimes \sigma_1 \otimes \cdots \otimes \sigma_4 \in X_4$ for $\sigma_i \in Q_t$. The product of A_4 and S_0 which is any generator of $H^0(Q_t, {}_\varphi\Lambda)$ are obtained by the following maps:

$$\begin{aligned} \Lambda \otimes \Lambda &\xrightarrow{\alpha_4^{-1} \otimes \alpha_0^{-1}} \mathrm{Hom}_\Lambda(Y_4, {}_\varphi\Lambda) \otimes \mathrm{Hom}_\Lambda(Y_0, {}_\varphi\Lambda) \\ &\xrightarrow{(\Delta_Y)^{\#}_{4,0}} \mathrm{Hom}_\Lambda\left(Y_4, {}_\varphi\Lambda \otimes {}_\varphi\Lambda\right) \\ &\xrightarrow{\mathrm{Hom}(\mathrm{id}_{Y_4}, \mu)} \mathrm{Hom}_\Lambda\left(Y_4, {}_\varphi\Lambda\right) \\ &\xrightarrow{\alpha_4} \Lambda, \end{aligned}$$

where α_4 denotes the isomorphism $\mathrm{Hom}_\Lambda\left(Y_4, {}_\varphi\Lambda\right) \xrightarrow{\sim} \Lambda$ stated in Section 2, and so on. From (3.1) and (3.2), we have

$$\begin{aligned} &\alpha_4 \cdot \mathrm{Hom}(\mathrm{id}_{Y_4}, \mu) \cdot (\Delta_Y)^{\#}_{4,0} \cdot \alpha_4^{-1} \otimes \alpha_0^{-1}(A_4 \otimes S_0) \\ &\quad = \left(\alpha_4^{-1}(A_4) \cdot u_4 \cdot v_4(1)\right) \left(\alpha_0^{-1}(S_0) \cdot u_0([\cdot])\right), \end{aligned}$$

where u_i and v_j denote the chain transformations $X_i \to Y_i$ and $Y_j \to X_j$, respectively. Since u_0 is the identity map on Λ (see [HaSa, Proposition 2]) and $u_4 \cdot v_4$ induces the identity map on $H^4(Q_t, {}_\varphi\Lambda)$, the product of A_4 and S_0 corresponds to the multiplication in Λ. This completes the proof. $\square$

Finally, we compute the relations in degree 4. These are also obtained by the method similar to the propositions above.

Proposition 5. *The following equations hold in $H^4(Q_t, {}_\varphi\Lambda)$ for the generators $(A_\alpha)_2, (A_\beta)_2, (B_\alpha)_2, (B_\beta)_2, (C_i)_2$ $(i = 1, 2, \ldots, t-1), D_2, E_2$ of $H^2(Q_t, {}_\varphi\Lambda)$:*

(i) $4tA_4 = 4tA_4B_0 = 2tA_4(C_i)_0 = 4A_4D_0 = 4A_4E_0 = 0.$

(ii) $(A_\alpha)_2{}^2 = (A_\alpha)_2\,(B_\alpha)_2 = (A_\alpha)_2\,(C_i)_2 = 0,$
$(A_\alpha)_2\,(A_\beta)_2 = 2tA_4,\ (A_\alpha)_2\,(B_\beta)_2 = 2tA_4B_0,$
$(A_\alpha)_2\,D_2 = 2A_4D_0,\ (A_\alpha)_2\,E_2 = 2A_4E_0.$

(iii) $(B_\alpha)_2\,(A_\beta)_2 = 2tA_4B_0,\ (B_\alpha)_2\,(B_\beta)_2 = 2tA_4,$
$(B_\alpha)_2{}^2 = (B_\alpha)_2\,(C_i)_2 = 0,$
$(B_\alpha)_2\,D_2 = 2A_4D_0,\ (B_\alpha)_2\,E_2 = 2A_4E_0.$

(iv) $$(A_\beta)_2{}^2 = \begin{cases} 2tA_4 & (t \equiv 0 \mod 4) \\ 0 & (t \equiv 2 \mod 4), \end{cases}$$
$$(A_\beta)_2\,(B_\beta)_2 = \begin{cases} 2tA_4B_0 & (t \equiv 0 \mod 4) \\ 0 & (t \equiv 2 \mod 4), \end{cases}$$
$$(A_\beta)_2\,(C_i)_2 = tA_4\,(C_i)_0\,,$$
$$(A_\beta)_2\,D_2 = \begin{cases} 0 & (t \equiv 0 \mod 4) \\ 2A_4D_0 & (t \equiv 2 \mod 4), \end{cases}$$
$$(A_\beta)_2\,E_2 = \begin{cases} 2A_4E_0 & (t \equiv 0 \mod 4) \\ 0 & (t \equiv 2 \mod 4). \end{cases}$$

(v) $$(B_\beta)_2{}^2 = \begin{cases} 2tA_4 & (t \equiv 0 \mod 4) \\ 0 & (t \equiv 2 \mod 4), \end{cases}$$
$$(B_\beta)_2\,(C_i)_2 = tA_4\,(C_{t-i})_0\,,$$
$$(B_\beta)_2\,D_2 = \begin{cases} 0 & (t \equiv 0 \mod 4) \\ 2A_4D_0 & (t \equiv 2 \mod 4), \end{cases}$$
$$(B_\beta)_2\,E_2 = \begin{cases} 2A_4E_0 & (t \equiv 0 \mod 4) \\ 0 & (t \equiv 2 \mod 4). \end{cases}$$

(vi) $$(C_i)_2D_2 = \begin{cases} 2A_4D_0 & (i\ \textit{even}) \\ 2A_4E_0 & (i\ \textit{odd}), \end{cases} \quad (C_i)_2E_2 = \begin{cases} 2A_4E_0 & (i\ \textit{even}) \\ 2A_4D_0 & (i\ \textit{odd}), \end{cases}$$
$$(C_i)_2(C_j)_2 = A_4(U_{i+j} - U_{i-j}),$$
$$\textit{where } U_k = \begin{cases} (C_{-k})_0 & (-t < k < 0) \\ 2A_0 & (k = 0) \\ (C_k)_0 & (0 < k < t) \\ 2B_0 & (k = t) \\ (C_{2t-k})_0 & (t < k < 2t). \end{cases}$$

(vii) $$D_2{}^2 = -tA_4 + tA_4B_0 + t\sum_{l=1}^{\frac{t}{2}-1} A_4(C_{2l})_0,$$

$$D_2E_2 = t\sum_{l=0}^{\frac{t}{2}-1} A_4(C_{2l+1})_0.$$

(viii) $$E_2{}^2 = -tA_4 + tA_4B_0 + t\sum_{l=1}^{\frac{t}{2}-1} A_4(C_{2l})_0.$$

Proof. See Propositions 1 and 4 for (i). The other relations in degree 4 are obtained by means of the following homomorphisms:

$$\begin{aligned} \Lambda^2 \otimes \Lambda^2 &\xrightarrow{\alpha_2^{-1}\otimes\alpha_2^{-1}} \mathrm{Hom}_\Lambda(Y_2, {}_\varphi\Lambda) \otimes \mathrm{Hom}_\Lambda(Y_2, {}_\varphi\Lambda) \\ &\xrightarrow{(\Delta_Y)^{\#}_{2,2}} \mathrm{Hom}_\Lambda(Y_4, {}_\varphi\Lambda \otimes {}_\varphi\Lambda) \\ &\xrightarrow{\mathrm{Hom}(\mathrm{id}_{Y_4},\mu)} \mathrm{Hom}_\Lambda(Y_4, {}_\varphi\Lambda) \\ &\xrightarrow{\alpha_4} \Lambda, \end{aligned}$$

where α_2 denotes the isomorphism $\mathrm{Hom}_\Lambda(Y_2, {}_\varphi\Lambda) \xrightarrow{\sim} \Lambda^2$ stated in Section 2, and so on. In this proof below, we use the notation $\mapsto$ for the elementwise map by the natural map

$$\Lambda \otimes \Lambda \simeq \mathrm{Hom}_\Lambda(Y_4, {}_\varphi\Lambda \otimes {}_\varphi\Lambda) \xrightarrow{\mathrm{Hom}(\mathrm{id}_{Y_4},\mu)} \mathrm{Hom}_\Lambda(Y_4, {}_\varphi\Lambda) \simeq \Lambda.$$

(ii) The products of $(A_\alpha)_2$ and the generators of $H^2(Q_t, {}_\varphi\Lambda)$: Using Lemma 3 and the above maps, we have the following:

$$\left(\alpha_2^{-1}((A_\alpha)_2) \otimes \alpha_2^{-1}((A_\alpha)_2)\right)\Big((\Delta_Y)_{2,2}(1)\Big) = 0 \mapsto 0,$$

$$\left(\alpha_2^{-1}((A_\alpha)_2) \otimes \alpha_2^{-1}((A_\beta)_2)\right)\Big((\Delta_Y)_{2,2}(1)\Big) = \sum_{k=0}^{2t-1} 1 \otimes 1 \mapsto 2t,$$

$$\left(\alpha_2^{-1}((A_\alpha)_2) \otimes \alpha_2^{-1}((C_i)_2)\right)\Big((\Delta_Y)_{2,2}(1)\Big) = 0 \mapsto 0,$$

$$\left(\alpha_2^{-1}((A_\alpha)_2) \otimes \alpha_2^{-1}(D_2)\right)\Big((\Delta_Y)_{2,2}(1)\Big) = \sum_{k=0}^{2t-1} 1 \otimes x^{2k-2}y \mapsto 2My,$$

$$\left(\alpha_2^{-1}((A_\alpha)_2) \otimes \alpha_2^{-1}(E_2)\right)\Big((\Delta_Y)_{2,2}(1)\Big) = \sum_{k=0}^{2t-1} 1 \otimes x^{2k-1}y \mapsto 2Mxy.$$

By Proposition 3, note that $(B_\alpha)_2 = B_0(A_\alpha)_2$ and $(B_\beta)_2 = B_0(A_\beta)_2$ hold. Hence we have $(A_\alpha)_2{}^2 = (A_\alpha)_2(B_\alpha)_2 = (A_\alpha)_2(C_i)_2 = 0$, $(A_\alpha)_2(A_\beta)_2 =$

$2tA_4,\ (A_\alpha)_2\,(B_\beta)_2 = 2tA_4B_0,\ (A_\alpha)_2\,D_2 = 2A_4D_0,\ (A_\alpha)_2\,E_2 = 2A_4E_0$.
(iii) The products of $(B_\alpha)_2$ and the generators of $H^2(Q_{t,\varphi}\Lambda)$: Using (ii) above and Propositions 2 and 3, we have the following:

$$\begin{aligned}
(B_\alpha)_2\,(A_\beta)_2 &= B_0\,(A_\alpha)_2\,(A_\beta)_2 = 2tA_4B_0,\\
{(B_\alpha)_2}^2 &= B_0\,(A_\alpha)_2\,(B_\alpha)_2 = 0,\\
(B_\alpha)_2\,(B_\beta)_2 &= {B_0}^2\,(A_\alpha)_2\,(A_\beta)_2 = 2tA_4,\\
(B_\alpha)_2\,(A_\beta)_2 &= B_0\,(A_\alpha)_2\,(C_i)_2 = 0,\\
(B_\alpha)_2\,D_2 &= 2B_0\,(A_\alpha)_2\,D_2 = 2A_4D_0,\\
(B_\alpha)_2\,E_2 &= 2B_0\,(A_\alpha)_2\,E_2 = 2A_4E_0.
\end{aligned}$$

(iv) The products of $(A_\beta)_2$ and the generators of $H^2(Q_{t,\varphi}\Lambda)$: Similarly the calculations are given as follows:

$$\begin{aligned}
&\left(\alpha_2^{-1}((A_\beta)_2)\otimes\alpha_2^{-1}((A_\beta)_2)\right)\left((\Delta_Y)_{2,2}\,(1)\right)\\
&= -t\otimes 1 - \sum_{k=0}^{2t-1} 1\otimes 1 - \sum_{k=t+1}^{2t-1} t\otimes 1\\
&\mapsto -(t^2+2t),\\
&\left(\alpha_2^{-1}((A_\beta)_2)\otimes\alpha_2^{-1}((C_i)_2)\right)\left((\Delta_Y)_{2,2}\,(1)\right)\\
&= t\otimes x^i - t\otimes x^{-i} + \sum_{k=0}^{2t-1} 1\otimes x^{-i}\\
&\mapsto t(x^i + x^{-i}),\\
&\left(\alpha_2^{-1}((A_\beta)_2)\otimes\alpha_2^{-1}(D_2)\right)\left((\Delta_Y)_{2,2}\,(1)\right)\\
&= -t\otimes x^{-2}y - \sum_{k=t+1}^{2t-1} t\otimes x^{2k-2}y\\
&\mapsto -tMy,\\
&\left(\alpha_2^{-1}((A_\beta)_2)\otimes\alpha_2^{-1}(E_2)\right)\left((\Delta_Y)_{2,2}\,(1)\right)\\
&= -t\otimes x^{-1}y - \sum_{k=0}^{2t-1} 1\otimes x^{2k-1}y - \sum_{k=t+1}^{2t-1} t\otimes x^{2k-1}y\\
&\mapsto -(t+2)Mxy.
\end{aligned}$$

Hence we have the following:

$$(A_\beta)_2{}^2 = \begin{cases} 2tA_4 & (t \equiv 0 \mod 4) \\ 0 & (t \equiv 2 \mod 4), \end{cases}$$
$$(A_\beta)_2\,(B_\beta)_2 = B_0 (A_\beta)_2{}^2 = \begin{cases} 2tA_4B_0 & (t \equiv 0 \mod 4) \\ 0 & (t \equiv 2 \mod 4), \end{cases}$$
$$(A_\beta)_2\,(C_i)_2 = tA_4\,(C_i)_0\,,$$
$$(A_\beta)_2\,D_2 = \begin{cases} 0 & (t \equiv 0 \mod 4) \\ 2A_4D_0 & (t \equiv 2 \mod 4), \end{cases}$$
$$(A_\beta)_2\,E_2 = \begin{cases} 2A_4E_0 & (t \equiv 0 \mod 4) \\ 0 & (t \equiv 2 \mod 4). \end{cases}$$

(v) The products of $(B_\beta)_2$ and the generators of $H^2(Q_t, {}_\varphi\Lambda)$: By (iv) above and Propositions 2 and 3, we have the following:

$$(B_\beta)_2{}^2 = B_0{}^2 (A_\beta)_2{}^2 = \begin{cases} 2tA_4 & (t \equiv 0 \mod 4) \\ 0 & (t \equiv 2 \mod 4), \end{cases}$$
$$(B_\beta)_2\,(C_i)_2 = B_0\,(A_\beta)_2\,(C_i)_2 = tA_4\,(C_{t-i})_0\,,$$
$$(B_\beta)_2\,D_2 = B_0\,(A_\beta)_2\,D_2 = \begin{cases} 0 & (t \equiv 0 \mod 4) \\ 2A_4D_0 & (t \equiv 2 \mod 4), \end{cases}$$
$$(B_\beta)_2\,E_2 = B_0\,(A_\beta)_2\,E_2 = \begin{cases} 2A_4E_0 & (t \equiv 0 \mod 4) \\ 0 & (t \equiv 2 \mod 4). \end{cases}$$

(vi) The products of $(C_i)_2$ and the generators of $H^2(Q_t, {}_\varphi\Lambda)$: Since

$$\begin{aligned}
&\big(\alpha_2^{-1}((C_i)_2) \otimes \alpha_2^{-1}((C_j)_2)\big)\Big((\Delta_Y)_{2,2}\,(1)\Big) \\
&= (x^i - x^{-i}) \otimes x^j - (x^i - x^{-i}) \otimes x^{-j} \\
&\mapsto (x^{i+j} + x^{-(i+j)}) - (x^{i-j} + x^{j-i}), \\
&\big(\alpha_2^{-1}((C_i)_2) \otimes \alpha_2^{-1}(D_2)\big)\Big((\Delta_Y)_{2,2}\,(1)\Big) \\
&= -(x^i - x^{-i}) \otimes x^{-2}y + \sum_{k=0}^{2t-1} x^i \otimes x^{2k-2}y - \sum_{k=t+1}^{2t-1} (x^i - x^{-i}) \otimes x^{2k-2}y
\end{aligned}$$

$$
\begin{aligned}
&\mapsto (x^i + x^{-i})My,\\
&\left(\alpha_2^{-1}((C_i)_2) \otimes \alpha_2^{-1}(E_2)\right)\left((\varDelta_Y)_{2,2}(1)\right)\\
&= -(x^i - x^{-i}) \otimes x^{-1}y + \sum_{k=0}^{2t-1} x^i \otimes x^{2k-1}y - \sum_{k=t+1}^{2t-1} (x^i - x^{-i}) \otimes x^{2k-1}y\\
&\mapsto (x^i + x^{-i})Mxy,
\end{aligned}
$$

we have the following:

$$
(C_i)_2(C_j)_2 = A_4(U_{i+j} - U_{i-j}),
$$

$$
(C_i)_2D_2 = \begin{cases} 2A_4D_0 & (i \text{ even}) \\ 2A_4E_0 & (i \text{ odd}), \end{cases} \quad (C_i)_2E_2 = \begin{cases} 2A_4E_0 & (i \text{ even}) \\ 2A_4D_0 & (i \text{ odd}). \end{cases}
$$

(vii) The products of D_2 and the generators of $H^2(Q_t, {}_\varphi\varLambda)$: Since

$$
\begin{aligned}
&\left(\alpha_2^{-1}(D_2) \otimes \alpha_2^{-1}(D_2)\right)\left((\varDelta_Y)_{2,2}(1)\right)\\
&= -My \otimes x^{-2}y + \sum_{k=0}^{2t-1} x^{2k-2}y \otimes x^{2k-2}y - \sum_{k=t+1}^{2t-1} My \otimes x^{2k-2}y\\
&\mapsto 2tx^t - tM,\\
&\left(\alpha_2^{-1}(D_2) \otimes \alpha_2^{-1}(E_2)\right)\left((\varDelta_Y)_{2,2}(1)\right)\\
&= -My \otimes x^{-1}y - \sum_{k=t+1}^{2t-1} My \otimes x^{2k-1}y\\
&\mapsto -tMx,
\end{aligned}
$$

we have ${D_2}^2 = -tA_4 + tA_4B_0 + t\sum_{l=1}^{\frac{t}{2}-1} A_4(C_{2l})_0$ and $D_2E_2 = t\sum_{l=0}^{\frac{t}{2}-1} A_4(C_{2l+1})_0$.

(viii) The products of E_2 and the generators of $H^2(Q_t, {}_\varphi\varLambda)$: Since

$$
\begin{aligned}
&\left(\alpha_2^{-1}(E_2) \otimes \alpha_2^{-1}(E_2)\right)\left((\varDelta_Y)_{2,2}(1)\right)\\
&= -Mxy \otimes x^{-1}y - \sum_{k=0}^{2t-1} x^{2k-1}y \otimes x^{2k-1}y - \sum_{k=t+1}^{2t-1} Mxy \otimes x^{2k-1}y\\
&\mapsto -tM - 2tx^t,
\end{aligned}
$$

we have ${E_2}^2 = -tA_4 + tA_4B_0 + t\sum_{l=1}^{\frac{t}{2}-1} A_4(C_{2l})_0$. This completes the proof of Proposition 5. □

3.2. The case t odd.

In this subsection, we consider the case t odd. The calculation of the multiplicative structure is similar to Section 3.1.

First of all, we calculate the products of the generators of $H^0(Q_t, {}_{\varphi}\Lambda)$. By Proposition 1, we take the generators as follows:

$$\begin{aligned} &A_0 = 1,\ B_0 = x^t, \\ &(C_i)_0 = x^i + x^{-i} \quad (i = 1, 2, \ldots, t-1), \\ &D_0 = My,\ E_0 = Mxy. \end{aligned}$$

Note that A_0 is the identity element of the cohomology ring $H^*(Q_t, {}_{\varphi}\Lambda)$.

Proposition 6. *The following equations hold in $H^0(Q_t, {}_{\varphi}\Lambda)$ for the generators $A_0, B_0, (C_i)_0$ $(i = 1, 2, \ldots, t-1), D_0, E_0$ of $H^0(Q_t, {}_{\varphi}\Lambda)$:*

$$\begin{gathered} {B_0}^2 = A_0,\ B_0(C_i)_0 = (C_{t-i})_0,\ B_0D_0 = E_0,\ B_0E_0 = D_0, \\ (C_i)_0D_0 = \begin{cases} 2D_0 & (i \text{ even}) \\ 2E_0 & (i \text{ odd}), \end{cases} \quad (C_i)_0E_0 = \begin{cases} 2E_0 & (i \text{ even}) \\ 2D_0 & (i \text{ odd}), \end{cases} \\ {D_0}^2 = {E_0}^2 = tB_0 + t\sum_{l=1}^{\frac{t-1}{2}} (C_{2l-1})_0,\ D_0E_0 = tA_0 + t\sum_{l=1}^{\frac{t-1}{2}} (C_{2l})_0, \\ (C_i)_0(C_j)_0 = U_{i+j} + U_{i-j}, \\ \text{where } U_k = \begin{cases} (C_{-k})_0 & (-t < k < 0) \\ 2A_0 & (k = 0) \\ (C_k)_0 & (0 < k < t) \\ 2B_0 & (k = t) \\ (C_{2t-k})_0 & (t < k < 2t). \end{cases} \end{gathered}$$

Proof. Straightforward. (*cf.* Proposition 2.) □

Next, we compute the relations in degree 2. By Proposition 1, we take the generators of $H^2(Q_t, {}_{\varphi}\Lambda)$ as follows:

$$A_2 = \left(\frac{t-1}{2}, 1\right),\ B_2 = \left(\frac{t-1}{2}x^t, x^t\right),$$

$$(C_i)_2 = \left(x^i, 0\right) \quad (i = 1, 2, \dots, t-1),$$
$$D_2 = (y, y),\ E_2 = (0, xy).$$

The following Lemma 5 is useful for the proof of Proposition 7:

Lemma 5. *For $i = 1, 2, \dots, t-1$, the following particular equations hold in $H^2(Q_t, {}_\varphi\Lambda)$:*

$$\begin{aligned}
&\text{(i)} && (1, 0) = 2A_2,\ (x^t, 0) = 2B_2,\\
&\text{(ii)} && (0, 1) = (2-t)A_2,\ (0, x^t) = (2-t)B_2,\\
&\text{(iii)} && \left(x^{t+i}, 0\right) = -\left(C_{t-i}\right)_2,\\
&\text{(iv)} && \left(0, x^i + x^{-i}\right) = t\left(C_i\right)_2,\\
&\text{(v)} && \left(x^t y, x^t y\right) = -E_2,\\
&\text{(vi)} && \left(Mx^j y, 0\right) = \begin{cases} 2D_2 & (j \text{ even}) \\ 2E_2 & (j \text{ odd}), \end{cases}\\
&\text{(vii)} && \left((x^i + x^{-i})y, (x^i + x^{-i})y\right) = \begin{cases} 2D_2 & (i \text{ even}) \\ 2E_2 & (i \text{ odd}), \end{cases}\\
&\text{(viii)} && (0, My) = -tD_2.
\end{aligned}$$

Proof. In this proof, we use the notation introduced in Section 2.1. The proof is similar to Lemma 4.
(i), (ii): These are immediate from (2.7).
(iii), (iv): By (2.7), the following equations hold in $H^2(Q_t, {}_\varphi\Lambda)$:

$$\left(x^{t+i}, 0\right) = \left(x^{t-i} + x^{i-t}, 0\right) - \left(x^{t-i}, 0\right) = -\left(C_{t-i}\right)_2,$$
$$\left(0, x^i + x^{-i}\right) = t(x^i, 0) - t\left(x^i + x^{-i}, 0\right) + \left(tx^{-i}, x^i + x^{-i}\right) = t\left(C_i\right)_2.$$

(v), (vi): By Lemma 2 (i), (ii) and (2.7), the following equations hold in $H^2(Q_t, {}_\varphi\Lambda)$:

$$\left(x^t y, x^t y\right) = \boldsymbol{a}_t = -\boldsymbol{a}_1 + \sum_{k=0}^{\frac{t-3}{2}} \boldsymbol{b}_{2k+1} + \boldsymbol{b}_{2t+1} = -E_2,$$
$$(My, 0) = -\boldsymbol{a}_{2t-2} = 2\boldsymbol{a}_0 - \sum_{k=1}^{t-1} \boldsymbol{b}_{2k} - \boldsymbol{b}_{2t} = 2D_2,$$
$$(Mxy, 0) = \boldsymbol{a}_{2t+1} = -2\boldsymbol{a}_1 + \boldsymbol{b}_{2t+1} = 2E_2.$$

(vii), (viii): Note that the following equations hold:

$$\big((x^i+x^{-i})y,(x^i+x^{-i})y\big)=\begin{cases}2\boldsymbol{a}_1-\boldsymbol{b}_{2t-1} & (i=1)\\ \boldsymbol{a}_{2t-2}-\boldsymbol{b}_{2t-2}+\boldsymbol{b}_{2t} & (i=2)\\ 2\boldsymbol{a}_1-\boldsymbol{b}_{2t-i}-\boldsymbol{b}_{2t+1}+\boldsymbol{b}'_{i-2} & (i\ \text{odd}, i\geqslant 3)\\ \boldsymbol{a}_{2t-2}-\boldsymbol{b}_{2t-i}+\boldsymbol{b}'_{i-2} & (i\ \text{even}, i\geqslant 4),\end{cases}$$

$$(0,My)=\boldsymbol{a}_{t-1}+((t-1)/2)\boldsymbol{a}_{2t-2}+\boldsymbol{b}_{2t}+\sum_{k=1}^{\frac{t-3}{2}}\boldsymbol{b}'_{2k}.$$

By Lemma 2 (iv), (v) and (vi), $\boldsymbol{b}'_i$ $(i=1,2,\dots,t-1)$ are expressed by linear combinations of $4\boldsymbol{a}_1,\boldsymbol{b}_1,\boldsymbol{b}_2,\dots,\boldsymbol{b}_{2t-2},\boldsymbol{b}_{2t},\boldsymbol{b}_{2t+1}$. Applying Lemma 2 (i), (2.7) and (vi) of this lemma to the above equations, we have the results. This completes the proof. □

Proposition 7. *The following equations hold in $H^2(Q_t, {}_\varphi\Lambda)$ for the generators $A_0, B_0, (C_i)_0$ $(i=1,2,\dots,t-1), D_0, E_0$ of $H^0(Q_t,{}_\varphi\Lambda)$ and the generators $A_2, B_2, (C_i)_2$ $(i=1,2,\dots,t-1), D_2, E_2$ of $H^2(Q_t,{}_\varphi\Lambda)$:*

(i) $4A_2=4B_2=2t(C_i)_2=4D_2=4E_2=0.$

(ii) $B_0A_2=B_2,\ B_0B_2=A_2,\ B_0(C_i)_2=-(C_{t-i})_2,$
$B_0D_2=-E_2,\ B_0E_2=-D_2.$

(iii) $(C_i)_0A_2=t(C_i)_2,\ (C_i)_0B_2=t(C_{t-i})_2,$

$$(C_i)_0D_2=\begin{cases}2D_2 & (i\ \textit{even})\\ 2E_2 & (i\ \textit{odd}),\end{cases}\quad (C_i)_0E_2=\begin{cases}2E_2 & (i\ \textit{even})\\ 2D_2 & (i\ \textit{odd}),\end{cases}$$

$(C_i)_0(C_j)_2=V_{i+j}+V_{j-i},$

$$\textit{where } V_k=\begin{cases}-(C_{-k})_2 & (-t<k<0)\\ 2A_2 & (k=0)\\ (C_k)_2 & (0<k<t)\\ 2B_2 & (k=t)\\ -(C_{2t-k})_2 & (t<k<2t).\end{cases}$$

(iv) $D_0A_2=-D_2,\ D_0B_2=E_2,\ D_0(C_i)_2=\begin{cases}2D_2 & (i\ \textit{even})\\ 2E_2 & (i\ \textit{odd}),\end{cases}$

$$D_0D_2=\begin{cases}-B_2+t\displaystyle\sum_{l=1}^{\frac{t-1}{2}}(C_{2l-1})_2 & (t\equiv 1 \mod 4)\\ B_2+t\displaystyle\sum_{l=1}^{\frac{t-1}{2}}(C_{2l-1})_2 & (t\equiv 3 \mod 4),\end{cases}$$

$$D_0E_2 = \begin{cases} A_2 + t\sum_{l=1}^{\frac{t-1}{2}}(C_{2l})_2 & (t \equiv 1 \mod 4) \\ -A_2 + t\sum_{l=1}^{\frac{t-1}{2}}(C_{2l})_2 & (t \equiv 3 \mod 4). \end{cases}$$

(v) $E_0A_2 = E_2,\ E_0B_2 = -D_2,\ E_0(C_i)_2 = \begin{cases} 2E_2 & (i\ \mathit{even}) \\ 2D_2 & (i\ \mathit{odd}), \end{cases}$

$$E_0D_2 = \begin{cases} -A_2 + t\sum_{l=1}^{\frac{t-1}{2}}(C_{2l})_2 & (t \equiv 1 \mod 4) \\ A_2 + t\sum_{l=1}^{\frac{t-1}{2}}(C_{2l})_2 & (t \equiv 3 \mod 4), \end{cases}$$

$$E_0E_2 = \begin{cases} B_2 + t\sum_{l=1}^{\frac{t-1}{2}}(C_{2l-1})_2 & (t \equiv 1 \mod 4) \\ -B_2 + t\sum_{l=1}^{\frac{t-1}{2}}(C_{2l-1})_2 & (t \equiv 3 \mod 4). \end{cases}$$

Proof. See Proposition 1 for (i). The other calculations are similar to Proposition 3.
(ii) The products of B_0 and the generators of $H^2(Q_t, {}_\varphi\Lambda)$: By similar calculations to Proposition 3, we have the following:

$$\begin{aligned} B_0A_2 &= \left(\frac{t-1}{2}x^t, x^t\right) = B_2, \\ B_0B_2 &= \left(\frac{t-1}{2}, 1\right) = A_2, \\ B_0\,(C_i)_2 &= \left(x^{t+i}, 0\right) = -\,(C_{t-i})_2 \quad \text{(by Lemma 5 (iii))}, \\ B_0D_2 &= (x^ty, x^ty) = -E_2 \quad \text{(by Lemma 5 (v))}, \\ B_0E_2 &= -{B_0}^2D_2 = -D_2 \quad \text{(by Proposition 6)}. \end{aligned}$$

(iii) The products of $(C_i)_0$ and the generators of $H^2(Q_t, {}_\varphi\Lambda)$: Similarly we have the following:

$$(C_i)_0\,A_2 = \left(\frac{t-1}{2}(x^i + x^{-i}), x^i + x^{-i}\right) = t\,(C_i)_2$$

$$\text{(by (2.7) and Lemma 5 (iv))},$$

$$
\begin{aligned}
&(C_i)_0\, B_2 = (C_{t-i})_0\, A_2 = t\,(C_{t-i})_2 \quad \text{(by Proposition 6)},\\
&(C_i)_0\, D_2 = \big((x^i + x^{-i})y, (x^i + x^{-i})y\big) = \begin{cases} 2D_2 & (i \text{ even}) \\ 2E_2 & (i \text{ odd}) \end{cases}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 5 (vii))},\\
&(C_i)_0\, E_2 = -\,(C_{t-i})_0\, D_2 = \begin{cases} 2E_2 & (i \text{ even}) \\ 2D_2 & (i \text{ odd}) \end{cases} \quad \text{(by Proposition 6)}.
\end{aligned}
$$

Note that the following equation holds in $H^2(Q_t, {}_{\varphi}\Lambda)$ using Lemma 5 (i) and (iii):

$$
(x^k, 0) = \begin{cases} 2A_2 & (k = 0) \\ (C_k)_2 & (0 < k < t) \\ 2B_2 & (k = t) \\ -(C_{2t-k})_2 & (t < k < 2t). \end{cases}
$$

Checking all cases for $i + j$ and $j - i$, we have $(C_i)_0\,(C_j)_2 = (x^{i+j} + x^{j-i}, 0) = V_{i+j} + V_{j-i}$.
(iv) The products of D_0 and the generators of $H^2(Q_t, {}_{\varphi}\Lambda)$: Similarly we have the following:

$$
\begin{aligned}
&D_0 A_2 = \left(\frac{t-1}{2} My, My\right) = -D_2 \quad \text{(by Lemma 5 (vi) and (viii))},\\
&D_0 B_2 = B_0 D_0 A_2 = E_2 \quad \text{(by Proposition 6)},\\
&D_0\,(C_i)_2 = (Mx^{-i}y, 0) = \begin{cases} 2D_2 & (i \text{ even}) \\ 2E_2 & (i \text{ odd}) \end{cases} \quad \text{(by Lemma 5 (vi))},\\
&D_0 D_2 = (Mx, Mx) = -tB_2 + t\sum_{l=1}^{\frac{t-1}{2}} (C_{2l-1})_2\\
&\qquad\qquad \text{(by (2.7), Lemma 5 (i), (ii) and (iv))},\\
&D_0 E_2 = -B_0 D_0 D_2 = tA_2 + t\sum_{l=1}^{\frac{t-1}{2}} (C_{2l})_2 \qquad \text{(by Proposition 6)}.
\end{aligned}
$$

(v) The products of E_0 and the generators of $H^2(Q_t, {}_{\varphi}\Lambda)$: Using the above calculations and Proposition 6, we have the following:

$$
\begin{aligned}
E_0 A_2 &= B_0 D_0 A_2 = -B_0 D_2 = E_2,\\
E_0 B_2 &= B_0 D_0 B_2 = B_0 E_2 = -D_2,
\end{aligned}
$$

$$E_0\,(C_i)_2 = B_0D_0\,(C_i)_2 = \begin{cases} 2E_2 & (i \text{ even}) \\ 2D_2 & (i \text{ odd}), \end{cases}$$

$$E_0D_2 = -{B_0}^2D_0E_2 = -tA_2 + t\sum_{l=1}^{\frac{t-1}{2}}(C_{2l})_2,$$

$$E_0E_2 = -{B_0}^2D_0D_2 = tB_2 + t\sum_{l=1}^{\frac{t-1}{2}}(C_{2l-1})_2.$$

This completes the proof of Proposition 7. □

In the following, we put $A_4 = 1$ which is a generator of $H^4(Q_t, {}_\varphi\Lambda)$. By Remark 1, A_4 is an invertible element in the complete cohomology ring $\hat{H}^*(Q_t, {}_\varphi\Lambda)$. Then we have the following proposition:

Proposition 8. *The generators of $H^4(Q_t, {}_\varphi\Lambda)$ are expressed by the product of A_4 and the generators $A_0, B_0, (C_i)_0$ $(i = 1, 2, \dots, t-1), D_0, E_0$ of $H^0(Q_t, {}_\varphi\Lambda)$, that is, the following equations hold in $H^4(Q_t, {}_\varphi\Lambda)$:*

$$x^t = A_4B_0,\ x^i + x^{-i} = A_4\,(C_i)_0 \quad (i = 1, 2, \dots, t-1),$$
$$My = A_4D_0,\ Mxy = A_4E_0.$$

Proof. The proof is similar to Proposition 4. □

Finally, we compute the relations in degree 4.

Proposition 9. *The following equations hold in $H^4(Q_t, {}_\varphi\Lambda)$ for the generators $A_2, B_2, (C_i)_2\,(i = 1, 2, \dots, t-1), D_2, E_2$ of $H^2(Q_t, {}_\varphi\Lambda)$:*

(i) $4tA_4 = 4tA_4B_0 = 2tA_4(C_i)_0 = 4A_4D_0 = 4A_4E_0 = 0.$

(ii) $${A_2}^2 = \begin{cases} tA_4 & (t \equiv 1 \mod 4) \\ -tA_4 & (t \equiv 3 \mod 4), \end{cases}$$

$$A_2B_2 = \begin{cases} tA_4B_0 & (t \equiv 1 \mod 4) \\ -tA_4B_0 & (t \equiv 3 \mod 4), \end{cases}$$

$A_2(C_i)_2 = tA_4(C_i)_0,\ A_2D_2 = -A_4D_0,\ A_2E_2 = A_4E_0.$

(iii) $${B_2}^2 = \begin{cases} tA_4 & (t \equiv 1 \mod 4) \\ -tA_4 & (t \equiv 3 \mod 4), \end{cases}$$

$B_2(C_i)_2 = tA_4(C_{t-i})_0,\ B_2D_2 = -A_4E_0,\ B_2E_2 = A_4D_0.$

(iv) $$(C_i)_2D_2 = \begin{cases} 2A_4D_0 & (i \textit{ even}) \\ 2A_4E_0 & (i \textit{ odd}), \end{cases} \quad (C_i)_2E_2 = \begin{cases} 2A_4E_0 & (i \textit{ even}) \\ 2A_4D_0 & (i \textit{ odd}), \end{cases}$$

$$(C_i)_2(C_j)_2 = A_4(U_{i+j} - U_{i-j}),$$

$$\textit{where } U_k = \begin{cases} (C_{-k})_0 & (-t < k < 0) \\ 2A_0 & (k = 0) \\ (C_k)_0 & (0 < k < t) \\ 2B_0 & (k = t) \\ (C_{2t-k})_0 & (t < k < 2t). \end{cases}$$

(v) $${D_2}^2 = tA_4B_0 + t\sum_{l=1}^{\frac{t-1}{2}} A_4(C_{2l-1})_0,\ D_2E_2 = -tA_4 + t\sum_{l=1}^{\frac{t-1}{2}} A_4(C_{2l})_0.$$

(vi) $${E_2}^2 = tA_4B_0 + t\sum_{l=1}^{\frac{t-1}{2}} A_4(C_{2l-1})_0.$$

Proof. See Propositions 1 and 8 for (i). The other calculations are similar to Proposition 5. In this proof also, we use the notation $\mapsto$ for the elementwise map by the natural map

$$\Lambda \otimes \Lambda \simeq \mathrm{Hom}_\Lambda(Y_4, {}_\varphi\Lambda \otimes {}_\varphi\Lambda) \xrightarrow{\mathrm{Hom}(\mathrm{id}_{Y_4},\mu)} \mathrm{Hom}_\Lambda(Y_4, {}_\varphi\Lambda) \simeq \Lambda.$$

(ii) The products of A_2 and the generators of $H^2(Q_t, {}_\varphi\Lambda)$: Using Lemma 3, we have the following:

$$\begin{aligned}
&\left(\alpha_2^{-1}(A_2) \otimes \alpha_2^{-1}(A_2)\right)\left((\Delta_Y)_{2,2}(1)\right) \\
&= \sum_{k=0}^{2t-1} 1 \otimes (t-1)/2 - t \otimes 1 - \sum_{k=0}^{2t-1} (1-(t-1)/2) \otimes 1 - \sum_{k=t+1}^{2t-1} t \otimes 1 \\
&\mapsto t^2 - 4t, \\
&\left(\alpha_2^{-1}(A_2) \otimes \alpha_2^{-1}((C_i)_2)\right)\left((\Delta_Y)_{2,2}(1)\right) \\
&= t \otimes x^i - t \otimes x^{-i} + \sum_{k=0}^{2t-1} 1 \otimes x^{-i} \\
&\mapsto t(x^i + x^{-i}), \\
&\left(\alpha_2^{-1}(A_2) \otimes \alpha_2^{-1}(D_2)\right)\left((\Delta_Y)_{2,2}(1)\right) \\
&= -t \otimes x^{-2}y + \sum_{k=0}^{2t-1} (t-1)/2 \otimes x^{2k-2}y - \sum_{k=t+1}^{2t-1} t \otimes x^{2k-2}y \\
&\mapsto -My,
\end{aligned}$$

By Proposition 7, note that $B_2 = B_0A_2$ and $E_2 = -B_0D_2$ hold. Hence we have

$$\begin{aligned}
{A_2}^2 &= \begin{cases} tA_4 & (t \equiv 1 \mod 4) \\ -tA_4 & (t \equiv 3 \mod 4), \end{cases} \\
A_2B_2 &= \begin{cases} tA_4B_0 & (t \equiv 1 \mod 4) \\ -tA_4B_0 & (t \equiv 3 \mod 4), \end{cases} \\
A_2\left(C_i\right)_2 &= tA_4\left(C_i\right)_0, \\
A_2D_2 &= -A_4D_0, \\
A_2E_2 &= A_4E_0.
\end{aligned}$$

(iii) The products of B_2 and the generators of $H^2(Q_t, {}_\varphi\Lambda)$: By (ii) above and Propositions 6 and 7, we have the following:

$$\begin{aligned}
{B_2}^2 &= (B_0A_2)^2 = {A_2}^2 = \begin{cases} tA_4 & (t \equiv 1 \mod 4) \\ -tA_4 & (t \equiv 3 \mod 4), \end{cases} \\
B_2\left(C_i\right)_2 &= B_0A_2\left(C_i\right)_2 = tA_4\left(C_{t-i}\right)_0, \\
B_2D_2 &= B_0A_2D_2 = -A_4E_0, \\
B_2E_2 &= B_0A_2E_2 = A_4D_0.
\end{aligned}$$

(iv) The products of $(C_i)_2$ and the generators of $H^2(Q_t, {}_\varphi\Lambda)$: By Proposition 7 and the similar calculations to Proposition 5 (vi), we have the following:

$$\begin{aligned}
(C_i)_2(C_j)_2 &= (x^{i+j} + x^{-(i+j)}) - (x^{i-j} + x^{j-i}) = A_4(U_{i+j} - U_{i-j}), \\
(C_i)_2D_2 &= (x^i + x^{-i})My = \begin{cases} 2A_4D_0 & (i \text{ even}) \\ 2A_4E_0 & (i \text{ odd}), \end{cases} \\
(C_i)_2E_2 &= -B_0(C_i)_2D_2 = (C_{t-i})_2D_2 = \begin{cases} 2A_4E_0 & (i \text{ even}) \\ 2A_4D_0 & (i \text{ odd}). \end{cases}
\end{aligned}$$

(v) The products of D_2 and the generators of $H^2(Q_t, {}_\varphi\Lambda)$: By Proposition 7 and the similar calculations to Proposition 5 (vii), we have the following:

$$\begin{aligned}
{D_2}^2 &= 2tx^t - tMx = tA_4B_0 + t\sum_{l=1}^{\frac{t-1}{2}} A_4(C_{2l-1})_0, \\
D_2E_2 &= -B_0{D_2}^2 = -tA_4 + t\sum_{l=1}^{\frac{t-1}{2}} A_4(C_{2l})_0.
\end{aligned}$$

(vi) The products of E_2 and the generators of $H^2(Q_t, {}_\varphi\Lambda)$: Using (v) above and Propositions 6 and 7, we have

$$E_2{}^2 = (B_0D_2)^2 = D_2{}^2 = tA_4B_0 + t\sum_{l=1}^{\frac{t-1}{2}} A_4(C_{2l-1})_0.$$

This completes the proof of Proposition 9. □

3.3. Main Theorem.

We will state the ring structure of $H^*(Q_t, {}_\varphi\Lambda)$ by summarizing Propositions 2 through 9 and Remark 1.

Theorem. *Let Q_t be the generalized quaternion group of order $4t$. We set $\Lambda = \mathbb{Z}Q_t$.*

(i) *If t is even, the Hochschild cohomology ring $H^*(Q_t, {}_\varphi\Lambda)(\simeq HH^*(\Lambda))$ is commutative, generated by the elements*

$$A_0,\ B_0,\ (C_i)_0\,,\ D_0,\ E_0 \in H^0(Q_t, {}_\varphi\Lambda),$$
$$(A_\alpha)_2\,,\ (A_\beta)_2\,,\ (B_\alpha)_2\,,\ (B_\beta)_2\,,\ (C_i)_2\,,\ D_2,\ E_2 \in H^2(Q_t, {}_\varphi\Lambda),$$
$$A_4 \in H^4(Q_t, {}_\varphi\Lambda),$$

for $i = 1, 2, \ldots, t-1$, where A_0 is the identity element. The relations are given by Table 1.

(ii) *If t is odd, the Hochschild cohomology ring $H^*(Q_t, {}_\varphi\Lambda)(\simeq HH^*(\Lambda))$ is commutative, generated by the elements*

$$A_0,\ B_0,\ (C_i)_0\,,\ D_0,\ E_0 \in H^0(Q_t, {}_\varphi\Lambda),$$
$$A_2,\ B_2,\ (C_i)_2\,,\ D_2,\ E_2 \in H^2(Q_t, {}_\varphi\Lambda),$$
$$A_4 \in H^4(Q_t, {}_\varphi\Lambda)$$

for $i = 1, 2, \ldots, t-1$, where A_0 is the identity element. The relations are given by Table 2.

Table 1. Cohomology ring $H^*(Q_{t,\varphi}\Lambda)$ for t even.

		B_0	$(C_j)_0$	D_0	E_0	$(A_\alpha)_2$	$(A_\beta)_2$	$(B_\alpha)_2$	$(B_\beta)_2$	$(C_j)_2$	D_2	E_2
B_0		A_0	$(C_{t-j})_0$	D_0	E_0	$(B_\alpha)_2$	$(B_\beta)_2$	$(A_\alpha)_2$	$(A_\beta)_2$	$-(C_{t-j})_2$	$-D_2$	$-E_2$
$(C_i)_0$	(i even)		$U_{i+j}+U_{i-j}$	$2D_0$	$2E_0$	0	$t(C_i)_2$	0	$t(C_{t-i})_2$	$V_{i+j}+V_{j-i}$	$2D_2$	$2E_2$
	(i odd)			$2E_0$	$2D_0$						$2E_2$	$2D_2$
D_0	$t\equiv 0\ (4)$			S_+	S'	$2D_2$	0	$2D_2$	0	$2D_2$ (j even)	T'	T''
	$t\equiv 2\ (4)$						$2D_2$		$2D_2$	$2E_2$ (j odd)		
E_0	$t\equiv 0\ (4)$				S_+	$2E_2$	$2E_2$	$2E_2$	$2E_2$	$2E_2$ (j even)	T''	T
	$t\equiv 2\ (4)$						0		0	$2D_2$ (j odd)		
${}_2(A_\alpha)_2$						0	$2tA_4$	0	$2tA_4B_0$	0	$2A_4D_0$	$2A_4E_0$
${}_2(A_\beta)_2$	$t\equiv 0\ (4)$						$2tA_4$	$2tA_4B_0$	$2tA_4B_0$	$tA_4(C_j)_0$	0	$2A_4E_0$
	$t\equiv 2\ (4)$						0		0		$2A_4D_0$	0
${}_2(B_\alpha)_2$								0	$2tA_4$	0	$2A_4D_0$	$2A_4E_0$
${}_2(B_\beta)_2$	$t\equiv 0\ (4)$								$2tA_4$	$tA_4(C_{t-j})_0$	0	$2A_4E_0$
	$t\equiv 2\ (4)$								0		$2A_4D_0$	0
${}_{2t}(C_i)_2$	(i even)									$A_4(U_{i+j}-U_{i-j})$	$2A_4D_0$	$2A_4E_0$
	(i odd)										$2A_4E_0$	$2A_4D_0$
${}_4D_2$											A_4S_-	A_4S'
${}_4E_2$												A_4S_-
${}_{4t}A_4$		${}_{4t}(A_4B_0)$	${}_{2t}(A_4(C_j)_0)$	${}_4(A_4D_0)$	${}_4(A_4E_0)$							

${}_lW_m$ means that l is the order of $W_m \in H^m(Q_{t,\varphi}\Lambda)$ as a $\mathbb{Z}$-module.

$$S_\pm := \pm tA_0 + tB_0 + t\sum_{l=1}^{\frac{t}{2}-1}(C_{2l})_0,\ S' := t\sum_{l=0}^{\frac{t}{2}-1}(C_{2l+1})_0,\ T := (A_\beta)_2 + (B_\beta)_2 + t\sum_{l=1}^{\frac{t}{2}-1}(C_{2l})_2,\ T' := (A_\alpha)_2 + (B_\alpha)_2 + T,\ T'' := t\sum_{l=0}^{\frac{t}{2}-1}(C_{2l+1})_2,$$

$$U_k = \begin{cases} (C_{-k})_0 & (-t<k<0) \\ 2A_0 & (k=0) \\ (C_k)_0 & (0<k<t) \\ 2B_0 & (k=t) \\ (C_{2t-k})_0 & (t<k<2t), \end{cases} \quad V_k = \begin{cases} -(C_{-k})_2 & (-t<k<0) \\ (A_\alpha)_2 & (k=0) \\ (C_k)_2 & (0<k<t) \\ (B_\alpha)_2 & (k=t) \\ -(C_{2t-k})_2 & (t<k<2t). \end{cases}$$

Table 2. Cohomology ring $H^*(Q_t, {}_\varphi\Lambda)$ for t odd.

		B_0	$(C_j)_0$	D_0	E_0	A_2	B_2	$(C_j)_2$	D_2	E_2
B_0		A_0	$(C_{t-j})_0$	E_0	D_0	B_2	A_2	$-(C_{t-j})_2$	$-E_2$	$-D_2$
$(C_i)_0$	(i even) (i odd)		$U_{i+j}+U_{i-j}$	$2D_0$ $2E_0$	$2E_0$ $2D_0$	$t(C_i)_2$	$t(C_{t-i})_2$	$V_{i+j}+V_{j-i}$	$2D_2$ $2E_2$	$2E_2$ $2D_2$
D_0	$t \equiv 1\ (4)$ $t \equiv 3\ (4)$			S'	S	$-D_2$	E_2	$\begin{cases} 2D_2 & (j \text{ even}) \\ 2E_2 & (j \text{ odd}) \end{cases}$	$-T'$ T'	T $-T$
E_0	$t \equiv 1\ (4)$ $t \equiv 3\ (4)$				S'	E_2	$-D_2$	$\begin{cases} 2E_2 & (j \text{ even}) \\ 2D_2 & (j \text{ odd}) \end{cases}$	$-T$ T	T' $-T'$
${}_4A_2$	$t \equiv 1\ (4)$ $t \equiv 3\ (4)$					tA_4 $-tA_4$	tA_4B_0 $-tA_4B_0$	$tA_4(C_j)_0$	$-A_4D_0$	A_4E_0
${}_4B_2$	$t \equiv 1\ (4)$ $t \equiv 3\ (4)$						tA_4 $-tA_4$	$tA_4(C_{t-j})_0$	$-A_4E_0$	A_4D_0
${}_{2t}(C_i)_2$	(i even) (i odd)							$A_4(U_{i+j}-U_{i-j})$	$2A_4D_0$ $2A_4E_0$	$2A_4E_0$ $2A_4D_0$
${}_4D_2$									A_4S'	$-A_4S$
${}_4E_2$										A_4S'
${}_{4t}A_4$		${}_{4t}(A_4B_0)$	${}_{2t}(A_4(C_j)_0)$	${}_4(A_4D_0)$	${}_4(A_4E_0)$					

${}_lW_m$ means that l is the order of $W_m \in H^m(Q_t, {}_\varphi\Lambda)$ as a $\mathbb{Z}$-module.

$$S := tA_0 + t\sum_{l=1}^{\frac{t-1}{2}}(C_{2l})_0,\ S' := tB_0 + t\sum_{l=1}^{\frac{t-1}{2}}(C_{2l-1})_0,\ T := A_2 + t\sum_{l=1}^{\frac{t-1}{2}}(C_{2l})_2,\ T' := B_2 + t\sum_{l=1}^{\frac{t-1}{2}}(C_{2l-1})_2,$$

$$U_k = \begin{cases} (C_{-k})_0 & (-t < k < 0) \\ 2A_0 & (k = 0) \\ (C_k)_0 & (0 < k < t) \\ 2B_0 & (k = t) \\ (C_{2t-k})_0 & (t < k < 2t), \end{cases} \quad V_k = \begin{cases} -(C_{-k})_2 & (-t < k < 0) \\ 2A_2 & (k = 0) \\ (C_k)_2 & (0 < k < t) \\ 2B_2 & (k = t) \\ -(C_{2t-k})_2 & (t < k < 2t). \end{cases}$$

(*Note*: S', T, T', V_k are independent of the notation in Table 1.)

Example. Let $\Lambda = \mathbb{Z}Q_2$ for the quaternion group Q_2 of order 8. We give the table of multiplicative structure for the Hochschild cohomology ring $H^*(Q_{t}, {}_{\varphi}\Lambda)(\simeq (HH^*(\Lambda))$, which is the case of $t = 2$ in Table 1.

Table 3. Cohomology ring $H^*(Q_2, {}_{\varphi}\Lambda)$.

	B_0	$(C_1)_0$	D_0	E_0	$(A_\alpha)_2$	$(A_\beta)_2$	$(B_\alpha)_2$	$(B_\beta)_2$	$(C_1)_2$	D_2	E_2
B_0	A_0	$(C_1)_0$	D_0	E_0	$(B_\alpha)_2$	$(B_\beta)_2$	$(A_\alpha)_2$	$(A_\beta)_2$	$-(C_1)_2$	$-D_2$	$-E_2$
$(C_1)_0$		$2(A_0+B_0)$	$2E_0$	$2D_0$	0	$2(C_1)_2$	0	$2(C_1)_2$	$(A_\alpha)_2+(B_\alpha)_2$	$2E_2$	$2D_2$
D_0			$2(A_0+B_0)$	$2(C_1)_0$	$2D_2$	$2D_2$	$2D_2$	$2D_2$	$2E_2$	$(A_\alpha)_2+(B_\alpha)_2$ $+(A_\beta)_2+(B_\beta)_2$	$2(C_1)_2$
E_0				$2(A_0+B_0)$	$2E_2$	0	$2E_2$	0	$2D_2$	$2(C_1)_2$	$(A_\beta)_2+(B_\beta)_2$
${}_2(A_\alpha)_2$					0	$4A_4$	0	$4A_4B_0$	0	$2A_4D_0$	$2A_4E_0$
${}_2(A_\beta)_2$						0	$4A_4B_0$	0	$2A_4(C_1)_0$	$2A_4D_0$	0
${}_2(B_\alpha)_2$							0	$4A_4$	0	$2A_4D_0$	$2A_4E_0$
${}_2(B_\beta)_2$								0	$2A_4(C_1)_0$	$2A_4D_0$	0
${}_4(C_1)_2$									$2A_4(B_0-A_0)$	$2A_4E_0$	$2A_4D_0$
${}_4D_2$										$2A_4(B_0-A_0)$	$2A_4(C_1)_0$
${}_4E_2$											$2A_4(B_0-A_0)$
${}_8A_4$	${}_8(A_4B_0)$	${}_4(A_4(C_1)_0)$	${}_4(A_4D_0)$	${}_4(A_4E_0)$							

${}_lW_m$ means that l is the order of $W_m \in H^m(Q_2, {}_{\varphi}\Lambda)$ as a $\mathbb{Z}$-module.

Remark 2. A precise description of the cohomology ring $H^*(Q_t, \mathbb{Z})$ for $t \geqslant 2$ is given by [HaSa, Section 4]:

$$H^*(Q_t, \mathbb{Z})$$
$$= \begin{cases} \mathbb{Z}[A,B,C]/(2A, 2B, 4tC, A^2, B^2 - 2tC, AB - 2tC) & (t \equiv 0 \mod 4) \\ \mathbb{Z}[A,B,C]/(2A, 2B, 4tC, A^2, B^2, AB - 2tC) & (t \equiv 2 \mod 4) \\ \mathbb{Z}[X,Y]/(4X, 4tY, X^2 - tY) & (t \equiv 1 \mod 4) \\ \mathbb{Z}[X,Y]/(4X, 4tY, X^2 + tY) & (t \equiv 3 \mod 4), \end{cases}$$

where $\deg A = \deg B = \deg X = 2$ and $\deg C = \deg Y = 4$. If t is even, the monomorphism of the cohomology rings $H^*(Q_t, \mathbb{Z}) \to H^*(Q_t, {}_\varphi\Lambda)$ is induced by the map $A \mapsto (A_\alpha)_2, B \mapsto (A_\beta)_2, C \mapsto A_4$. So we may identify the generators of cohomology ring $H^*(Q_t, \mathbb{Z})$ with the subring of $H^*(Q_t, {}_\varphi\Lambda)$ generated by $(A_\alpha)_2, (A_\beta)_2, A_4$ (see Proposition 5 (ii)). If t is odd, this is induced by the map $X \mapsto A_2, Y \mapsto A_4$. So we may identify the generators of cohomology ring $H^*(Q_t, \mathbb{Z})$ with the subring of $H^*(Q_t, {}_\varphi\Lambda)$ generated by A_2 and A_4 (see Proposition 9 (ii)).

Acknowledgement

The author would like to express his gratitude to Professor K. Sanada and Professor T. Nozawa for valuable comments and many helpful suggestions. The author would also like to express his gratitude to the referee for valuable comments and many helpful suggestions which help to improve the original presentation of this paper.

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